The Automorphism Group of a Compact Hyperbolic Riemann Surface is Finite*

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The goal of today's talk is to prove the finiteness property of the automorphism group of a hyperbolic compact Riemann surface M. The word hyperbolic can be replaced with the condition that the topological genus of M g is greater or equal than 2. This is because every such Riemann surface admits a Riemannian metric g of constant -1 Gaussian curvature, i.e. a hyperbolic structure[1]. Such g will be called a hyperbolic metric. We formulate the above as the following theorem:

Theorem 1. *Let M be a compact Riemann surface of genus $g \geq 2$, then the automorphism group $Aut(M)$, which is group of biholomorphisms, or equivalently, the group of orientation preserving conformal automorphisms, is finite.*

This is a quite interesting phenomenon, since this result is false for Riemann surfaces of elliptic or parabolic type.

Example 0.1. *Suppose M is a compact Riemann surface of genus $g = 0$, then M is biholomorphic to $\mathbb{CP}^1$. Then $Aut(\mathbb{CP}^1) = \mathbb{P}SL_2(\mathbb{C})$, which is not finite. In this case, the Gaussian curvature K of M is positive.*

Example 0.2. *Suppose M is a compact Riemann surface of genus $g = 1$, then M is biholomorphic to $\mathbb{C}/\Lambda$, for some lattice Λ. Its automorphisms are in general translations, and square lattice and hexagonal lattice have addition symmetries from rotation by $90°$ and $60°$. In particular $Aut(M)$ is infinite as well. In this case, the Gaussian curvature K of M is zero.*

We shall present two proofs of theorem 1, one differential geometric, and the other topological. The second proof will only be sketched.

Recall that $g \geq 2$ implies that M has a hyperbolic metric g. We shall assume the fact that $\mathrm{Aut}(M) = \mathrm{Isom}^+(M, g)$. Let ∇ be the Levi-Civita connection, and R the Riemannian

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[1]This is a consequence of the uniformization theorem.

curvature tensor. Locally, the Ricci curvature tensor Ric is defined as

$$Ric_p(X,Y) := \sum_i R(X, e_i, e_i, Y),$$

where $p \in M$, $X \in T_pM$, and $\{e_i\}$ is a local orthonormal frame of the tangent bundle. From the symmetries of the Riemann curvature tensor, we see that the Ricci tensor is symmetric. We set $Ric_p(X) := Ric_p(X,X)$. In the case where M is a surface, we have

$$Ric_p(X) = K(p)g(X,X),$$

where $K(p)$ is a Gaussian curvature at p. Hence negative Gaussian curvature is equivalent to negative Ricci curvature in the case of surfaces. Therefore theorem1 follows from the following more general theorem due to Bochner[1]:

Theorem 2 (Bochner)**.** *Let (M^n, g) be a compact Riemannian manifold of dimension n, with negative definite Ricci curvature everywhere, then the isometry group of M is finite.*

Proof. It is well known that the isometry group of a Riemannian manifold has a Lie group structure with respect to the compact open topology in M, and if M is compact, so is $\mathrm{Isom}(M)$(cf.[2]). Thus it suffices to show that the connected component of $\mathrm{Isom}(M)$ which contains the identity, $\mathrm{Isom}^0(M)$, is just the identity. Suppose not, then there exists a smooth one parameter family of isometries φ_t with $\varphi_0 = id$. Then $\xi(t) := \frac{d\varphi_t}{dt}$ is a Killing vector field on M. Therefore $L_\xi g = 0$, where L_ξ is the Lie derivative along ξ. In local coordinates (U, x^i), we can rewrite this as:

$$g_{ij,k}\xi^k + g_{kj}\xi^k_{,i} + g_{ik}\xi^k_{,j} = 0, \forall\, i,j,k \tag{1}$$

where $g_{ij,k} = \frac{\partial g_{ij}}{\partial x^k}$, and $\xi = \xi^k \frac{\partial}{\partial x^k}$, $\xi^k_{,i} = \frac{\partial \xi^k}{\partial x^i}$. At the center p of a geodesic normal neighborhood, we have

$$g_{ij,k}(p) = 0, \Gamma^k_{ij}(p) = 0.$$

Therefore, at p, (1) becomes

$$\xi^j_{,i} + \xi^i_{,j} = 0, \forall\, i,j. \tag{2}$$

Using the metric g, we have an isomorphism $\Phi : TM \longrightarrow T^*M$, $v \mapsto g(v,\cdot)$. If $v = v^i \frac{\partial}{\partial x^i}$ locally, then $v^\flat := g(v,\cdot) = g_{ij}dx^i(v)dx^j = g_{ij}v^i dx^j$. Setting $\langle v, w\rangle = \langle v^\flat, w^\flat\rangle$, Φ is then an isometry. It is easy to check that (2) is equivalent to

$$(\nabla\xi^\flat)_{ij} = -(\nabla\xi^\flat)ji, \tag{3}$$

i.e.

$$(\nabla_j\xi^\flat)_i = -(\nabla_i\xi^\flat)_j, \tag{4}$$

abbreviating ∇_i as $\nabla_{\frac{\partial}{\partial x^i}}$. Here we use the same notation ∇ to denote the induced connection on the cotangent bundle. Now consider the smooth function $f(q) := |\xi(q)|^2 = |\xi^\flat(q)|^2$, the norm squared of ξ at $q \in M$. Since M is compact, f achieves its maximum at some point, say p. Fix a geodesic normal neighborhood (U, x^i) centered at p, we have:

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\begin{aligned}
& \sum_i \frac{1}{2} \frac{\partial^2}{(\partial x^i)^2} |\xi^\flat|^2(p) \leq 0 && \\
\Longrightarrow\ & \frac{\partial}{\partial i} \langle \nabla_i \xi^\flat, \xi^\flat \rangle (p) \leq 0 && \text{(metric compatibility)} \\
\Longrightarrow\ & |\nabla_i \xi^\flat|^2(p) + \langle \nabla_i \nabla_i \xi^\flat, \xi^\flat \rangle (p) \leq 0 && \\
\Longrightarrow\ & |\nabla_i \xi^\flat|^2(p) + (\nabla_i \nabla_i \xi^\flat)_j \cdot \xi^\flat_j(p) \leq 0 && (g_{ij}(p) = \delta_{ij}) \\
\Longrightarrow\ & |\nabla_i \xi^\flat|^2(p) + (\nabla_i \xi^\flat)_{j,i} \cdot \xi^\flat_j(p) \leq 0 && (\Gamma^k_{ij}(p) = 0) \\
\Longrightarrow\ & |\nabla_i \xi^\flat|^2(p) - (\nabla_j \xi^\flat)_{i,i} \cdot \xi^\flat_j(p) \leq 0 && \text{(by (4))} \\
\Longrightarrow\ & |\nabla_i \xi^\flat|^2(p) - (\nabla_i \nabla_j \xi^\flat)_i \cdot \xi^\flat_j(p) \leq 0 && (\Gamma^k_{ij}(p) = 0) \\
\Longrightarrow\ & |\nabla_i \xi^\flat|^2(p) - (\nabla_j \nabla_i \xi^\flat)_i \cdot \xi^\flat_j(p) - (R_{ij}\xi^\flat)_i \cdot \xi^\flat_j(p) \leq 0 && ([\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial x^j}] = 0) \\
\Longrightarrow\ & |\nabla_i \xi^\flat|^2(p) - (\nabla_i \xi^\flat)_{i,j} \cdot \xi^\flat_j(p) - R_{ijki}\xi^k \cdot \xi^j(p) \leq 0 && (g_{ij}(p) = \delta_{ij}) \\
\Longrightarrow\ & |\nabla_i \xi^\flat|^2(p) - 0 - Ric(\xi_p) \leq 0 && \\
\Longrightarrow\ & |\nabla_i \xi^\flat|^2(p) \leq Ric(\xi_p) \leq 0, && \\
\Longrightarrow\ & \xi(p) = 0 && \text{(Ricci tensor is negative definite)} \\
\Longrightarrow\ & \xi \equiv 0 && (p \text{ is the length maximum})
\end{aligned}
$$

Thus there exists no nontrivial Killing field on M, and $\mathrm{Isom}^0(M) = \{id\}$. □

Now we sketch the second proof. Let $\Omega^1(M)$ be the space of holomorphic one forms on M, which are forms that locally looks like $f(z)dz$, where f is a holomorphic function. Note that every holomorphic one form is closed, since locally

$$d(f(z)dz) = \frac{\partial f}{\partial z} dz \wedge dz + \frac{\partial f}{\partial \overline{z}} d\overline{z} \wedge dz = 0$$

due to holomorphicity. Moreover, no holomorphic one form is exact, suppose otherwise, then there exists $\omega \in \Omega^1(M)$ such that $\omega = df$, where $f : M \longrightarrow \mathbb{C}$ is necessarily a holomorphic function. But since M is compact, f must be constant, hence $\omega = 0$. With these two observations, we have an injection:

$$\Omega^1(M) \hookrightarrow H^1(X, \mathbb{C}).$$

Let $\overline{\Omega}^1(M)$ be the space of antiholomorphic one forms, that locally looks like $\overline{f(z)} d\overline{z}$, where f is a holomorphic function. Then conjugation defines a conjugate linear isomorphism from $\Omega^1(M)$ to $\overline{\Omega}^1(M)$, and $\Omega^1(M) \cap \overline{\Omega}^1(M) = \{0\}$. In fact, more is true:

Theorem 3 (Hodge theorem for Riemann surfaces)**.** $H^1(M,\mathbb{C}) = \Omega^1(M) \oplus \overline{\Omega}^1(M)$. *Note that* $H^1(X,\mathbb{C})$ *is a complex vector space of complex dimension* $2g$*, and can be viewed as* $H^1(X,\mathbb{R}) \oplus iH^1(X,\mathbb{R})$*.* g *is the genus of* M*.*

We shall assume this result. Suppose $\varphi \in \mathrm{Aut}(M)$, then φ acts on $\Omega^1(M)$ isomorphically via pull-back. Therefore we have

$$F : \mathrm{Aut}(M) \longrightarrow \mathrm{Aut}(\Omega^1(M)) \cong GL_g(\mathbb{C}) \supset U(g).$$

Fact 1: The image of F actually lands in the compact group $U(g)$.
Note that φ will also induce an isomorphism on integral homology $H^1(M,\mathbb{Z})$. Hence we have a map

$$\Phi : \mathrm{Aut}(X) \longrightarrow \mathrm{Aut}(H^1(M,\mathbb{Z})) \cong GL_{2g}(\mathbb{Z}).$$

Fact 2: Φ is an injection.
Therefore $\mathrm{Aut}(M)$ is a discrete subgroup of a compact group, hence must be finite.

Remark. *There is another way of proving theorem1. I will just briefly talk about the idea. Again we assume there exists a smooth one parameter family of isometries* $\{\varphi_t\}$*, and we get the Killing field* $\xi(t)$ *as usual. Then one can show that* ξ *has isolated zeros on* M*, hence finite since* M *is compact. Then the Hopf index theorem gives:*

$$\sum_p i_p\xi = \chi(M) < 0,$$

where the sum is taken over points p *such that* $\xi(p) = 0$*. For such* p*,* φ_t *fixes* p*. We can lift* φ_t *to a one parameter family of isometries* $\widetilde{\varphi}_t$ *on the upper half plane* $\mathbb{H}$*, and each* $\widetilde{\varphi}_t$ *has a fixed point. Without loss of generality, we can assume* $\widetilde{\varphi}_t$ *fixes* i*, hence must be an element of* $SO(2)$*. Therefore the index of* ξ *at that point* p *must have index* 1 *since elements of* $SO(2)$ *have determinant* 1*. This implies that*

$$\sum_p i_p\xi = \textit{number of fixed points} \geq 0,$$

unless $\xi \equiv 0$*.*

References

[1] S. Bochner; *Vectorfields and Ricci curvature*, **Bull. Amer. Math. Soc. 52** (1946), 776-797.

[2] S. Kobayashi and K. Nomizu; *Foundations of Differential Geometry*, Vol I. Interscience Tracts in Pure and Applied Mathematics No.15, John Wiley and Sons.