Applications of Jacobi Field Estimates on Topology and Curvature*

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Last time we proved the Fundamental Theorem of Morse Theory which states that:

Theorem 1. *Let M be a complete Riemannian manifold, and let $p, q \in M$ be two points which are not conjugate along any geodesic. Then $\Omega(M; p, q)$ has the homotopy type of a countable CW-complex which contains one cell of dimension λ for each geodesic from p to q of index λ.*

This gives a good topological description of the path space $\Omega(M; p, q)$ provided that p and q are non-conjugate along any geodesic. We first prove that such a pair of non-conjugate points exists. Recall that a smooth map $f : M \longrightarrow N$ between manifolds of the same dimension is critical at a point $p \in M$ if the include map df_p on tangent spaces has nontrivial kernel.

Theorem 2. *Let M be a complete manifold. $\exp_p v$ is conjugate to p along the geodesic γ_v from p to $\exp_p v$ if and only if $\exp_p : T_pM \longrightarrow M$ is critical at v, i.e., $(d\exp_p)_v$ has nontrivial kernel.*

Proof. Suppose that v is a critical point of $\exp_p$. Let $0 \neq X \in T_v(T_pM)$ be in the kernel of $(d\exp_p)_v : T_v(T_pM) \cong T_pM \longrightarrow T_{\exp_p v}M$. Let $u \mapsto v(u)$ be a smooth path in T_pM such that $v(0) = v$, and $v'(0) = X$. Then the map $\alpha : (u, t) \mapsto \exp_p tv(u) \in M$ defines a smooth variation through geodesics of the geodesic γ_v, which is given by $t \mapsto \exp_p tv$. Therefore the variational vector field $t \mapsto \frac{\partial}{\partial u}[\exp_p tv(u)]|_{u=0}$ is a Jacobi field $J(t)$ along γ_v. Note that

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$J(0) = 0$. Moreover,

$$J(1) = \frac{\partial}{\partial u}[\exp_p v(u)]|_{u=0} = (d\exp_p)_v(v'(0)) == (d\exp_p)_v(X) = 0. \qquad (1)$$

But $J(t)$ is not identically zero since

$$\begin{aligned}(D_{\frac{\partial}{\partial t}} J)(0) = \left(D_{\frac{\partial}{\partial t}} \left[\frac{\partial}{\partial u}[\exp_p tv(u)]|_{u=0} \right] \right)_{t=0} &= \left(D_{\frac{\partial}{\partial u}} \left[\frac{\partial}{\partial t}[\exp_p tv(u)]|_{t=0} \right] \right)_{u=0} \\ &= \left(D_{\frac{\partial}{\partial u}} \left[(d\exp_p)_0(v(u)) \right] \right)_{u=0} \\ &= \left(D_{\frac{\partial}{\partial u}} \left[v(u) \right] \right)_{u=0} \\ &= \frac{\partial v}{\partial u}|_{u=0} \\ &= X \\ &\neq 0.\end{aligned}$$

So there is a non-trivial Jacobi field along γ_v from p to $\exp_p v$, vanishing at these two points; hence p and $\exp_p v$ are conjugate along γ_v.
Conversely, suppose $\exp_p$ is non-singular at $v \in T_pM$, we can choose n linearly independent vectors $X_1, \cdots, X_n$ in $T_v(T_pM)$ so that $\{(d\exp_p)(X_i)|i = 1, \cdots n\}$ is a set of linearly independent vectors. Choose paths $u \mapsto v_i(u)$ in T_pM, $i = 1, \cdots, n$, such that $v_i(0) = v$, and $v_i'(0) = X_i$ for each i. We can construct n variations through geodesics $\alpha_1, \cdots, \alpha_n$ similarly as above. From them we obtain n Jacobi fields $J_1, \cdots, J_n$ along γ_v, vanishing at p. From (1) we see that $J_i(1) = (d\exp_p)_v(X_i)$, hence $J_1(1), \cdots, J_n(1)$ are linearly independent. Recall the dimension of the space of Jacobi field along γ_v vanishing at p is n. Therefore there exists no non-trivial Jacobi field J along γ_v that vanishes at both p and $\exp_p v$. □

Corollary 3. *Let $p \in M$. Then for almost all $q \in M$, p is not conjugate to q along any geodesic.*

Now we study the behavior of geodesics on manifolds with positive sectional curvature and negative sectional curvature. Recall that the sectional curvature of the plane spanned by the linearly independent tangent vectors $X = a^i\partial_i|_p, Y = b^j\partial_j|_p \in T_pM$ is defined as

$$K(X \wedge Y) := \frac{\langle R(X,Y)X, Y\rangle}{|X \wedge Y|^2} [1]$$

[1] This is Milnor's convention.

where $|X \wedge Y|^2 := \langle X, X\rangle \cdot \langle Y, Y\rangle - \langle X, Y\rangle^2$.

Lemma 1. *Suppose M is a manifold of nonpositive sectional curvature, then no two points of M are conjugate along any geodesic.*

Proof. Let γ be a geodesic with velocity vector field V, and let J be a Jacobi field along γ. We use J' to denote $\frac{DJ}{dt}$, the covariant derivative of J along γ. Then from the curvature hypothesis and the Jacobi equation:

$$\langle J'', J\rangle = -\langle R(V,J)V, J\rangle = -2K(V,J)|V \wedge J|^2 \geq 0$$

Therefore $\frac{d}{dt}\langle J', J\rangle = \langle J'', t\rangle + |J'|^2 \geq 0$. Thus the function $\langle J', J\rangle$ is monotone increasing, and strictly so if $J' \neq 0$. Suppose $J(0) = J(t_0) = 0$ for some $t_0 > 0$, then $\langle J', J\rangle(0) = \langle J', J\rangle(t_0) = 0$ as well. Hence $\langle J', J\rangle \equiv 0$ on $[0, t_0]$ by monotonicity. But this implies that $J'(0) = 0$. Since J is determined by $J(0)$ and $J'(0)$, which both vanish, $J \equiv 0$. This completes the proof. □

Theorem 4 (Cartan)**.** *Suppose M is a simply connected, complete Riemannian manifold, and has nonpositive sectional curvature. Then any two points of M are joined by a unique geodesic. Furthermore, M is diffeomorphic to $\mathbb{R}^n$.*

Proof. Since there are no conjugate points, it follows from the index theorem that every geodesic from p to q has index $\lambda = 0$. Thus Theorem 1 implies that $\Omega(M; p, q)$ has the homotopy type of a 0-dimensional CW-complex, with one vertex for each geodesic connecting p and q. Since M is simply connected, $\Omega(M, ; p, q)$ is thus connected, Hence $\Omega(M; p, q)$ has the homotopy type of a point. Hence there is precisely one geodesic from p to q. Therefore the exponential map $\exp_p : T_pM \longrightarrow M$ is one-to-one and onto, and is critical point free (hence a local diffeomorphism), thus $\exp_p$ is a global diffeomorphism. □

Remark. More generally, if M is complete, has nonpositive sectional curvature, but not simply connected, (e.g., M might be a flat torus $S^1 \times S^1$ or a compact surface of genus ≥ 2 with constant negative sectional curvature.) Then Theorem 4 applies to the universal covering $\widetilde{M}$ of M, where $\widetilde{M}$ is endowed with the pullback metric from M which is also complete, and has sectional curvature ≤ 0, since curvature and metric are purely local invariants. For details, we refer the reader to [1], Chapter 7, Exercise 2.

Corollary 5. *If M is a connected complete Riemannian manifold with non-positive sectional curvature, then the higher homotopy groups $\pi_i(M) = 0$ for all $i > 0$, and $\pi_1(M)$ is torsion free, i.e. there is no element of finite order other than the identity.*

Proof. Using the long exact sequence of homotopy groups of the fiber bundle $\widetilde{M} \longrightarrow M$ with discrete fibers, we see that $\pi_i(M) \cong \pi_i(\widetilde{M}) \cong \pi_i(\mathbb{R}^n)$ for all $i > 0$. Since $\widetilde{M}$ is now contractible, we have

$$H^\bullet(\pi_1(M)) \cong H^\bullet(M),$$

with coefficient any left $\mathbb{Z}[\pi_1(M)]$ module (i.e. a module that admits a $\pi_1(M)$ action). Here $H^\bullet(\pi_1(M))$ is the group cohomology computed using free resolutions. Suppose G is a non-trivial cyclic subgroup of $\pi_1(M)$, then we can construct a covering space $\widehat{M}$ over M such that $\pi_1(\widehat{M}) = G$. Hence

$$H^k(G) \cong H^k(\widehat{M}) = 0$$

for $k > \dim M$, since $\widetilde{M}$ is the universal covering of $\widehat{M}$ as well. However, it is a general fact that the group cohomology of a finite cyclic group are nonzero in arbitrarily high dimensions. This gives a contradiction. Therefore $\pi_1(M)$ is torsion free, and in particular, infinite. □

Remark. From the above corollary, we see that if M is a manifold whose universal covering is contractible, then it also follows that $\pi_1(M)$ must be torsion free, hence infinite. Thus the universal covering being contractible puts strong restrictions on the topology of M. This result is also known as the *Smith's Theorem*.

Now we will consider manifolds with positive Ricci curvature, which can be thought of as the average of sectional curvature over all the possible two planes.

Definition 1. *The Ricci curvature in the direction $X = \sum_i a^i|_p \in T_pM$ is defined as*

$$Ric_p(X, X) := \sum_{j,l} g^{jl} \langle R(X, \partial_j)X, \partial_l \rangle$$

Thus if we choose an orthonormal basis $e_1, \cdots, e_n$ of T_pM, then the Ricci curvature at p in the direction e_j is given by

$$\mathrm{Ric}_p(e_j, e_j) = \sum_{i=1}^{n-1} \langle R(e_j, e_i)e_j, e_i \rangle.$$

Theorem 6 (Myers)**.** *Suppose that the Ricci curvature satisfies* $Ric(U,U) \geq \frac{n-1}{r^2} > 0$ *for every unit tangent vector* U *at every point of* M*, where* r *is a positive constant. Then every geodesic of* M *of length* $> \pi r$ *contains conjugate points; and hence is not length minimizing.*

Proof. Let $\gamma : [0,1] \longrightarrow M$ be a geodesic of length L. Choose parallel vector fields $P_1, \cdots, P_n$ along γ which are orthonormal at one point and hence are orthonormal everywhere along γ. We assume that P_n points along γ so that

$$V := \frac{d\gamma}{dt} = LP_n, \quad \text{and } \frac{DP_i}{dt} = 0.^2$$

Let $W_i(t) := (\sin \pi t) P_i(t)$. Then from the second variation formula, we have

$$\frac{1}{2} E_{**}(W_i, W_i) = \int_0^1 (\sin \pi t)^2 \left(\pi^2 - L^2 \langle R(P_n, P_i) P_n, P_i \rangle \right) dt$$

Summing for $i = 1, \cdots, n-1$, we obtain:

$$\frac{1}{2} \sum_{i=1}^{n-1} E_{**}(W_i, W_i) = \int_0^1 (\sin \pi t)^2 \big((n-1)^2 \pi^2 - L^2 \text{Ric}(P_n, P_n) \big) dt.$$

Now if $\text{Ric}(P_n, P_n) \geq \frac{n-1}{r^2}$ and $L > \pi r$, then the above expression is negative, hence $E_{**}(W_i, W_i) < 0$ for some i. This implies that the index of γ is positive, and hence by the Index Theorem, γ contains conjugate points. It follows that γ is not length minimizing. □

Example. *If* M *is a sphere of radius* r*, then* M *has constant sectional curvature* $\frac{1}{r^2}$*. Hence its Ricci curvature is constantly* $\frac{n-1}{r^2}$*. Therefore by Theorem 6, every geodesic of length* $> \pi r$ *contains conjugate points.*

Corollary 7. *If* M *is a complete manifold with* $Ric(U,U) \geq \frac{n-1}{r^2} > 0$ *for all unit tangent vectors* U*, then* M *is compact with* $diam(M) \leq \pi r$*.*

Proof. If $p, q \in M$, let γ be a minimizing geodesic from p to q. Since M is complete, such geodesic exists. Then the length of γ must be $\leq \pi r$ by Theorem 6. Therefore all points have distance less or equal than πr. Since closed bounded sets in a complete manifold are compact, it follows that M itself is compact. □

[2] In fact, we can just let $P_n := \frac{V}{L}$, and set $P_1, \cdots, P_{n-1}$ to be orthonormal parallel vector fields that lie in the orthogonal complement of P_n.

This corollary also applies to the universal covering space $\widetilde{M}$ of M. Let $\pi : \widetilde{M} \longrightarrow M$ denote the covering projection. Since $\widetilde{M}$ is compact, it follows that each fiber of this covering map is finite, hence $\pi_1(M)$ is also finite since there is a one-to-one correspondence between $\pi^{-1}(p)$ and $\pi_1(M)$ for any $p \in M$ due to the fact the $\pi_1(M)$ acts on each fiber freely.

Theorem 8. *If M is a complete manifold, and if the Ricci tensor of M is everywhere positive definite, then the path space $\Omega(M; p, q)$ has the homotopy type of a CW-complex having only finitely many cells in each dimension.*

Proof. Since the space of unit tangent vectors U on M is compact, it follows that □

References

[1] Carmo, Do; *Riemannian Geometry.* Mathematics: Theory and Applications, Birkhäuser, 1992.

[2] Milnor, John; *Morse Theory.* Annals of Mathematics Studies No. 51, Princeton Unversity Press, 1973.