Quantitative canvas weave analysis using 2D synchrosqueezed transforms

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I. INTRODUCTION

QUANTITATIVE canvas weave analysis has many applications in art investigations of paintings, including dating, forensics, canvas rollmate identification [1]–[3]. Traditionally, canvas analysis is based on x-radiographs. Prior to serving as a painting canvas, a piece of fabric is coated with a priming agent; smoothing its surface makes this layer thicker between and thinner right on top of weave threads. These variations affect the x-ray absorption, making the weave pattern stand out in x-ray images of the finished painting. To characterize this pattern, it is customary to visually inspect small areas within the x-radiograph and count the number of horizontal and vertical weave threads; averages of these then estimate the overall canvas weave density. The tedium of this process typically limits its practice to just a few sample regions of the canvas. In addition, it does not capture more subtle information beyond weave density, such as thread angles or variations in the weave pattern. Application of signal processing techniques to art investigation are now increasingly used to develop computer-assisted canvas weave analysis tools.

In their pioneering work [4], Johnson *et al* developed an algorithm for canvas thread-counting based on windowed Fourier transforms; further developments in [5], [6] extract more information, such as thread angles and weave patterns. Successful applications to paintings of art historical interest include works by Vincent van Gogh [7], [8], Diego Velázquez [9], Johannes Vermeer [10], among others [11]–[16].

A more robust and automated analysis technique was later developed by Erdmann *et al* [17], based on autocorrelation and pattern recognition algorithms, requiring less human intervention. Unlike the Fourier-space based approach of [4], [17] uses only the real-space representation of the canvas. Likewise, [18] also uses real-space based features for canvas texture characterization.

We consider here a new automated analysis technique for quantitative canvas analysis, based on the $2D$ synchrosqueezed transforms recently developed in [19]–[21]. In this Fourier-space based method, the nonlinear synchrosqueezing procedure is applied to a phase-space representation of the image obtained by wavepacket or curvelet transforms. The method requires little human intervention; as shown below, it is very robust and offers fine scale weave density and thread angle information for the canvas. We compare our results with those of the celebrated method developed in [4]–[6].

We explain our model for x-radiography images for canvas analysis in Section II; the use and limitations of windowed Fourier transforms are discussed in Section III-A. Section III-B introduces the synchrosqueezed transform, with applications to quantitative canvas analysis; section IV presents various examples, applying our technique in art investigation.

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II. MODEL OF THE CANVAS WEAVE PATTERN IN X-RADIOGRAPHY

We denote by f the intensity of an x-radiograph of a painting; see Figure 1a for a (zoomed-in) example. Because x-rays penetrate deeply, the image consists of several components: the paint layer itself, primer, canvas (if the painting is on canvas or on wood panel overlaid with canvas), possibly a wood panel (if the painting is on wood), and sometimes extra slats (stretchers for a painting on canvas, or a cradle for a painting on wood, thinned and cradled according to earlier conservation practice.) This x-ray image may be affected by noise or artifacts of the acquisition process. We model the intensity function f as an additive superposition of the canvas contribution, denoted by $c(x)$, and a remainder, denoted by $p(x)$, that incorporates all the other components. Our approach to quantitative canvas analysis relies on a simple model for the x-ray image of the weave pattern in the “ideal” situation. Since it is produced by the interleaving of horizontal and vertical threads in a periodic fashion, a natural general model is

$$f(x) = c(x) + p(x) := a(x)S(2\pi N\phi(x)) + p(x). \quad (1)$$

In this expression, S is a periodic function on the square $[0, 2\pi)^2$, the details of which reflect the basic weave pattern of the canvas, e.g., whether it is a plain weave or perhaps a twill weave. This is a generalization of more specific assumptions used in the literature – for instance, in [4] plain weave canvas is modeled by taking for S a sum of sinusoidal functions in the x and y directions; in [6], more general weave patterns (in particular twill) are considered. The parameter N in (1) gives the averaged overall weave density of the canvas (in both directions). The function ϕ, which maps the image domain to $\mathbb{R}^2$, is a smooth deformation representing the local warping of the canvas; it contains information on local thread density, local thread angles, etc. The slowly varying function $a(x)$ takes into account the variation of the amplitude of the x-ray image of the canvas, possibly due to variation in illumination conditions.

In some cases, the x-ray image may fail to show canvas information in portions of the painting (e.g. when the paint layer dominates); the model (1) is then not uniformly valid. Because our analysis typically uses spatially localized information (analyzing the image patch by patch), this affects our results only locally: in those (small) portions of the image we have no good estimates for the canvas parameters. For simplicity, we assume in this exposition that (1) is valid for the whole image.

In our Fourier-space based canvas analysis we represent the weave pattern function S, periodic on $[0, 2\pi)^2$, in terms of its Fourier series,

$$c(x) = \sum_{n \in \mathbb{Z}^2} a(x) \widehat{S}(n) e^{2\pi i N n \cdot \phi(x)}. \tag{2}$$

This is a superposition of smoothly warped plane-waves with local wave vectors $N\nabla(n \cdot \phi(x))$. The idea of our analysis is to extract the function ϕ by exploiting that the Fourier coefficients $\{\widehat{S}(n)\}$ are dominated by a few leading terms.

III. Fourier-space based canvas analysis

A. Windowed Fourier transform

Although for canvas analysis $c(x)$ is really 2-dimensional, the main idea of our approach can be well understood using a $1D$ analog. Let's thus consider a one-dimensional function (2):

$$c(x) = \sum_{n \in \mathbb{Z}} a(x) \widehat{S}(n) e^{2\pi i N n \phi(x)} \tag{3}$$

where $x \in \mathbb{R}$. Because a and ϕ vary slowly with x, we can use Taylor expansions to approximate the function for x near x_0 as

$$c(x) \approx \sum_{n \in \mathbb{Z}} a(x_0) \widehat{S}(n) e^{2\pi i N n \phi(x_0)} e^{2\pi i N n (x - x_0)\phi'(x_0)}. \tag{4}$$

The right hand side of (4) is a superposition of complex exponentials with frequencies $Nn\phi'(x_0)$; these would stand out in a Fourier transform as peaks in the Fourier spectrum. Since the approximation is valid only near x_0, a windowed Fourier transform is needed, with envelope given by, e.g., a Gaussian centered at x_0 with width σ. We have then

$$\begin{aligned} W(x_0, k) &:= \frac{1}{\sqrt{2\pi\sigma^2}} \int e^{-2\pi i k (x - x_0)} e^{-(x-x_0)^2/2\sigma^2} c(x)\, \mathrm{d}x \\ &\approx \sum_{n \in \mathbb{Z}} a(x_0) \widehat{S}(n) e^{2\pi i N n \phi(x_0)} e^{-2\pi^2\sigma^2 [k - N n \phi'(x_0)]^2}. \end{aligned} \tag{5}$$

Instead of being sharply peaked, the spectrum of the windowed Fourier transform is thus "spread out" around the $Nn\phi'(x_0)$ – a manifestation of the well-known uncertainty principle in signal processing, with a trade-off w.r.t. the parameter σ: a larger σ reduces the "spreading" at the price of a larger error in the approximation (4), since the Gaussian is then correspondingly wider in the real space.

The method of [4], [6] uses the local maxima of the amplitude of the windowed Fourier transform to estimate the location of $\{N\nabla(n \cdot \phi(x_0))\}$ (the $2D$ analog of $Nn\phi'(x_0)$) for a selection of positions x_0 of the x-ray image (more precisely, local swatches are used instead of the Gaussian envelope, but the spirit is the same). For ideal signals, (5) shows that the maxima of the amplitude $|W(x_0, \cdot)|$ identify the dominating wave vectors in the Fourier-space, which are then used to extract information, including weave density and thread angles. Thread density is estimated by the length of the wave vectors, and the weave orientation is determined by the angles. This back-of-the-envelope calculation is reasonably precise when N is much larger than 1, resulting in a small $\mathcal{O}(1/N)$ error in the Taylor expansions, etc. In terms of the canvas analysis, this means that the inverse of the average thread density must be much smaller than the length scale of the variation of the canvas texture, typically on the scale of the size of the painting. This is essentially a high-frequency assumption, ensuring that stationary phase approximations can be applied in the time-frequency analysis. Details can be found in standard references of time-frequency analysis, e.g., the book [22].

In more complicated scenarios, in particular, when the x-ray signal corresponding to the canvas is heavily "contaminated" by the other parts of the painting, it is desirable to have more robust and refined analysis tools at hand than locating local maxima of the Fourier spectrum. The synchrosqueezed transforms are nonlinear time-frequency analysis tools developed for this purpose, and are hence suitable for canvas analysis in more challenging situations.

B. Synchrosqueezed transforms

The synchrosqueezed transforms, or more generally time-frequency reassignment techniques (see e.g., the recent review [23]), have been introduced to deal with the "loss of resolution" due to the uncertainty principle. Originally introduced in [24] for auditory signals, using a nonlinear squeezing of the time-frequency representation to gain sharpness of the time-frequency representation, the $1D$ synchrosqueezed wavelet transform was revisited and analyzed in [25]. For the application to canvas analysis, we rely on $2D$ extensions of the synchrosqueezing transforms based on wavepacket and curvelet transforms [19], [20]. This $2D$ synchrosqueezing transform has been applied to atomic-resolution crystal image analysis in [21]. In fact, the present algorithm for canvas analysis is adapted from the crystal image analysis developed in [21], a natural extension since canvas textures are very "crystalline" in nature, from an image analysis point of view.

Let us again use the $1D$ example to illustrate the main idea of synchrosqueezing; the technical details and the mathematical analysis of the algorithm can be found in [21]. The crucial observation is that the phase of the complex function $W(x, k)$, obtained from the windowed Fourier transform (5) contains information on the local frequency (i.e., the instantaneous frequency) of the signal. Indeed, for (x,k) such that k is close to $Nn\phi'(x)$, we have

$$w_f(x,k) := \frac{1}{2\pi}\Im\,\partial_x \ln W(x,k) = Nn\phi'(x) + o(N), \quad (6)$$

where $\Im(z)$ stands for the imaginary part of the complex number z. Motivated by this heuristic, the synchrosqueezed windowed Fourier transform "squeezes" the time-frequency spectrum by reassigning the amplitude at (x,k) to $(x, w_f(x,k))$ as

$$T(x,\xi) := \int |W(x,k)|^2 \delta(\xi - w_f(x,k))\,\mathrm{d}k. \quad (7)$$

This significantly enhances the sharpness of the time-frequency representation and yields a finer estimate of the local frequency of the signal.

This discussion generalizes readily to $2D$ signals. As the orientation of the local wave vector is important, wave packets and curvelets are used to enhance the resolution in the angular direction in the Fourier space. The local wave vector is estimated by the straightforward generalization of (6) to two dimension, and the squeezing of the coefficient (7) then gives a sharpened energy distribution on phase space:

$$T(x,\xi) \approx \sum_{n\in\mathbb{Z}^2} |a(x)|^2 \left|\widehat{S}(n)\right|^2 \delta(\xi - N\nabla(n\cdot\phi(x))) \quad (8)$$

in the sense of distribution. See [19]–[21] for more details, as well as an analysis of the method. The peaks of the synchrosqueezed spectrum T then provide estimates of the $N\nabla(n\cdot\phi(x))$, in turn determining the thread count and angle. Figure 1 illustrates the resulting spectrum of the synchrosqueezed transform compared with the windowed Fourier transform for a sample x-ray image from a canvas; it is apparent that after synchrosqueezing, the phase-space representation becomes much more compact. In particular, the red circles in Figure 1c indicate the (very precise!) locations of the dominating wave vectors, which determine the thread count and angle. The nice performance and the robustness of the synchrosqueezed transforms are supported by rigorous mathematical analysis in [19]–[21].

IV. Applications to art investigations

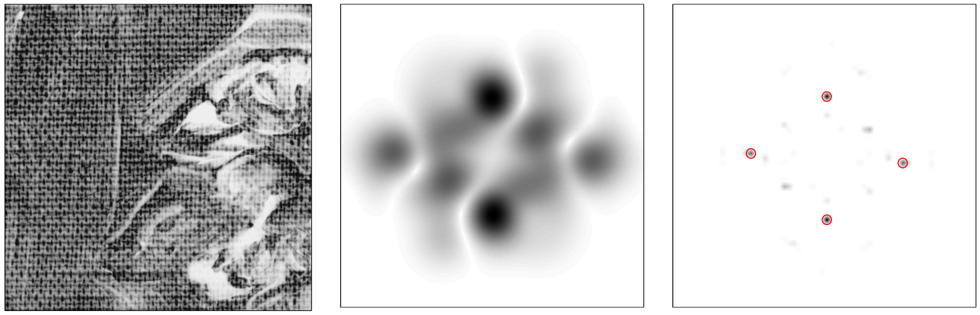
Let us now present some results of quantitative canvas analysis using $2D$ synchrosqueezed transforms. The algorithm is implemented in `Matlab`. The codes are open source and available for downloading as `SynLab` at http://web.stanford.edu/~haizhao/Codes.htm.

The first example is the painting F205 by Vincent van Gogh, the x-ray image of which is shown in Figure 2. The x-ray image is publicly available as part of the RKD dataset [26] provided by the Netherlands Institute for Art History. We choose this example because it was one of the first examples analyzed using the method based on the windowed Fourier transform; see [4, Figure 4] and also [6, Figure 6]. In Figure 3, the thread count and thread angle estimates are shown for horizontal and vertical threads. Comparing with the previous results in [4], [6], we observe that the general characteristics of the canvas agree quite well. For example, [6] reports average thread counts 13.3 threads/cm (horizontal) and 16.0 threads/cm (vertical), while our method obtains 13.239 threads/cm (horizontal) and 15.917 threads/cm (vertical). Compared to the earlier results, the current analysis gives a more refined spatial variation of the thread counts. In particular, it captures the oscillation of the thread count on a much finer scale. Detailed comparison using visual inspection confirms the presence of these fine details.

We next consider a painting of Vermeer, *Woman in Blue Reading a Letter* (L17), the x-ray image of which is also available as part of the RKD dataset [26]. The canvas analysis for Vermeer's paintings is considerably more challenging than that of van Gogh's [10], [27]. This can be understood by direct comparison of the x-ray images in Figures 4 and 2. The stretchers and nails significantly perturb the x-ray image for the Vermeer. Nevertheless the synchrosqueezed transform based canvas analysis performs quite well on the Vermeer example, as shown in Figure 5: although the thread count and angle estimate are affected by artifacts in the x-ray image, they still provide a detailed characterization of the canvas weave. Figure 6 shows a zoom-in for the x-ray image and the vertical thread angle map, further illustrating that the algorithm captures (and quantifies) the deviation in the vertical thread angle also recognizable by visual inspection.

To test the algorithm on a different type of canvas weave, we applied it to the x-ray image of Albert P. Ryder's *The Pasture*, a painting on twill canvas. Figure 7 shows the result for a portion of the canvas. The twill canvas pattern is clear on the zoomed-in x-ray image. The method is still able to capture fine scale features of the canvas; the admittedly higher number of artifacts is due to the increased difficulty to "read" a twill vs. a standard weave pattern, as well as a weaker canvas signal on the x-ray.

For our final example, we apply the synchrosqueezed transform based canvas analysis to *The Peruzzi Altarpiece* by Giotto di Bondone and his assistants. The altarpiece is in the collection of the North Carolina Museum of Art; see Figure 8 for the altarpiece as well as the x-ray images used in the analysis. This is a painting on wood panel, but the ground of traditional white gesso was applied over a coarsely woven fabric interlayer glued to a poplar panel. We carried out a canvas analysis on the fabric interlayer, likely a hand woven linen cloth. The results of a canvas analysis based on the synchrosqueezed transform are shown in Figure 10. This example is much more challenging than the previous ones, since the x-ray intensity contributed by the canvas is much weaker because the ground does not contain lead; see e.g. Figure 9, a detail of the x-ray image of the Christ panel. The canvas is barely visible, in sharp contrast to the x-ray image in Figure 6. All panels except the central Christ panel are cradled; the wood texture of these cradles interferes with the canvas pattern on the x-ray image, introducing an additional difficulty. This difficulty is reflected in our results: e.g., the vertical thread count for the central panel has much fewer artifacts than those of the other panels (see Figure 10). [In future work, we will explore carrying out a canvas analysis after signal-processing-based virtual cradle removal – see [28].]

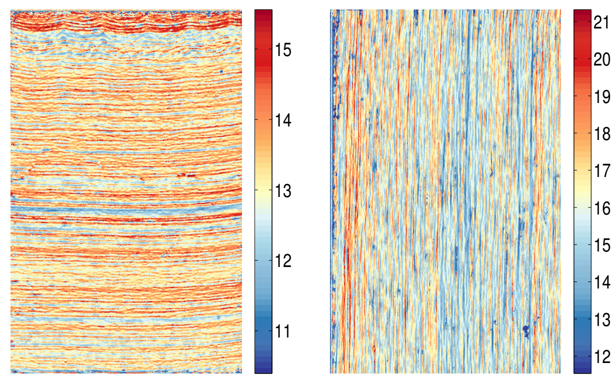


(a) A sample x-ray image (b) Spectrum of windowed Fourier transform at one spatial location (c) Spectrum of synchrosqueezed transform at one spatial location

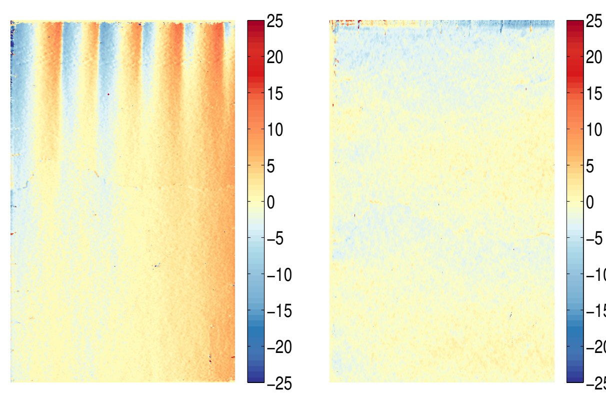
Fig. 1. (a) A sample swatch of the x-ray image. The canvas pattern is clearly recognizable with painting on top of the canvas; (b) The spectrum of the windowed Fourier transform as a function of frequency vector at one spatial location; (c) The spectrum of the synchrosqueezed transform at the same spatial location. After synchrosqueezing, the spectrum is much more localized. The red circles identify the peaks of the energy used to estimate local wave vectors.

Fig. 2. X-ray image of van Gogh's painting *Portrait of an Old Man with Beard*, 1885, Van Gogh Museum, Amsterdam (F205). Provided by Professor C. Richard Johnson through the RKD dataset [26].

One interesting ongoing art investigation debate concerning this altarpiece is the relative position of the panels of John the Baptist and Francis of Assisi. While the order shown in Figure 8, with Francis in the right-most position, and the Baptist second from right, is the most commonly accepted [29], there have been alternative arguments that the Francis panel should be instead placed next to the central panel. We wondered what ordering (if any) would be suggested by the canvas analysis. Under the assumption that the pieces of canvas are cut off consecutively from one larger piece of cloth, we investigate which arrangement provides the best matching. One plausible arrangement of the canvas is shown in Figure 11. Our analysis suggests that the canvas of the central panel should be rotated for 90 degrees clockwise to match with the other panels. (The larger height of the central panel, possibly exceeding the width of the cloth roll, may have necessitated this.) Moreover, a better matching is achieved if the canvas of the panel of the Baptist is flipped horizontally (in other words, flipped front to back). Given our results, it seems unlikely that the Francis-panel canvas would fit best to the left of the Baptist-panel canvas. A better, more precise result will be possible after virtual cradle removal. Of course, even a more conclusive canvas roll arrangement would provide at best an indirect hint for the relative position of the panels themselves; combined with other elements in a more exhaustive study, it can nevertheless play a role.



(a) Horizontal thread count (b) Vertical thread count



(c) Horizontal thread angle (d) Vertical thread angle

Fig. 3. The canvas analysis results of van Gogh's F205 using the synchrosqueezed transform: (a) and (b): thread count map of the horizontal and vertical threads; (c) and (d): the estimated thread angle. Compare with [6, Figure 6].

V. Conclusion

We apply $2D$ synchrosqueezed transforms to quantitative canvas weave analysis for art investigations. The synchrosqueezed transforms offer a sharpened phase-space representation of the x-ray image of the paintings, which yields fine scale characterization of thread count and thread angle of the canvas. We demonstrated the effectiveness of the method

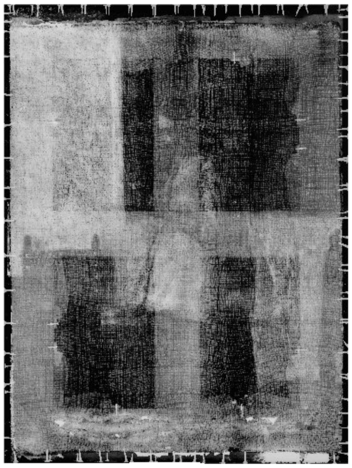
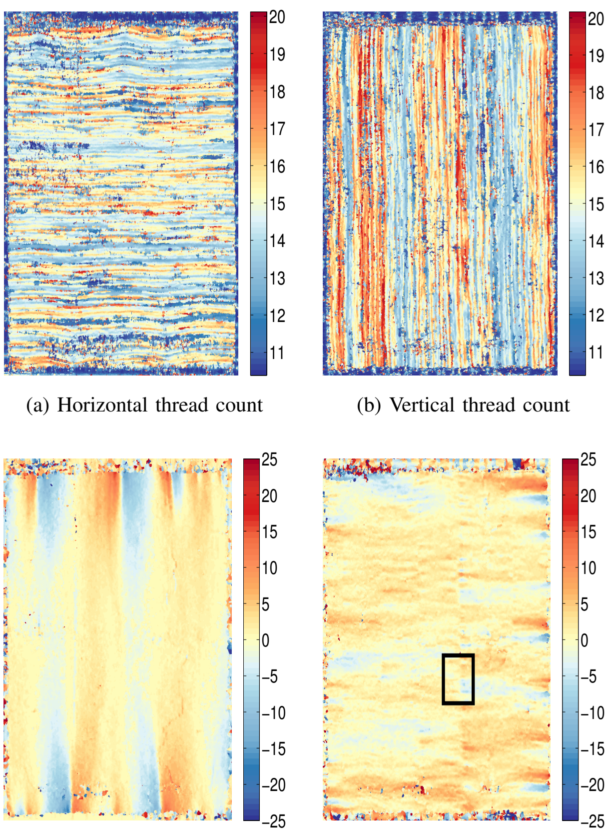


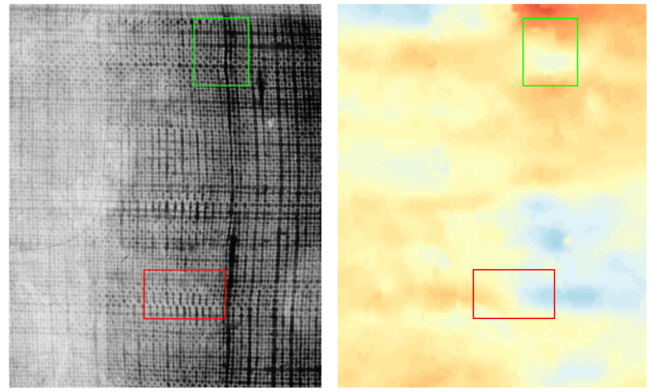
Fig. 4. X-ray image of Vermeer's painting *Woman in Blue Reading a Letter*, 1663-64, Rijksmuseum Amsterdam, Amsterdam (L17). X-ray image provided by Professor C. Richard Johnson through the RKD dataset [26].



(a) Horizontal thread count (b) Vertical thread count

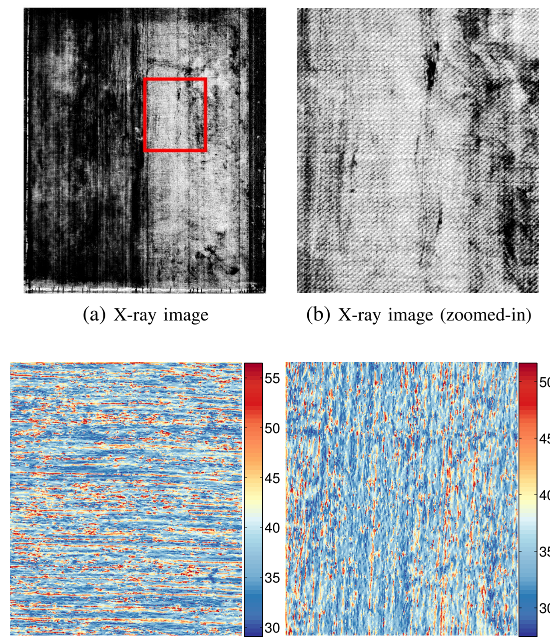
(c) Horizontal thread angle (d) Vertical thread angle

Fig. 5. Canvas analysis results of Vermeer's L17 using the synchrosqueezed transform: (a) and (b) are thread count map of the horizontal and vertical threads; (c) and (d) show the estimated thread angle. Average thread density is 14.407 threads/cm (horizontal) and 14.817 threads/cm (vertical). The boxed region of the vertical thread angle map (panel (d)) is shown, enlarged, in Figure 6.



(a) X-ray image (zoomed-in) (b) Vertical thread angle (zoomed-in)

Fig. 6. Details of the x-ray image and the corresponding vertical thread angle map for Vermeer's L17, highlighting two examples (boxed regions) of noticeable fine scale variation of the vertical thread angle, readily recognizable also by visual inspection of the corresponding zones in the x-ray image.



(a) X-ray image (b) X-ray image (zoomed-in)

(c) Horizontal thread count (d) Vertical thread count

Fig. 7. (a) X-ray image of Albert P. Ryder's *The Pasture*, 1880-85, North Carolina Museum of Art, Raleigh. The red-boxed region is enlarged in (b), with clearly recognizable twill canvas weave. (c) and (d) show the thread count maps correspond to the zoomed-in region shown in (b). Note the much higher thread counts than for plain weave canvas, typical for the finer threads used in twill weave.

on art works by van Gogh, Vermeer, and Ryder. The tool is applied to *The Peruzzi Altarpiece* by Giotto and his assistants, to provide insight into the issue of panel arrangement.

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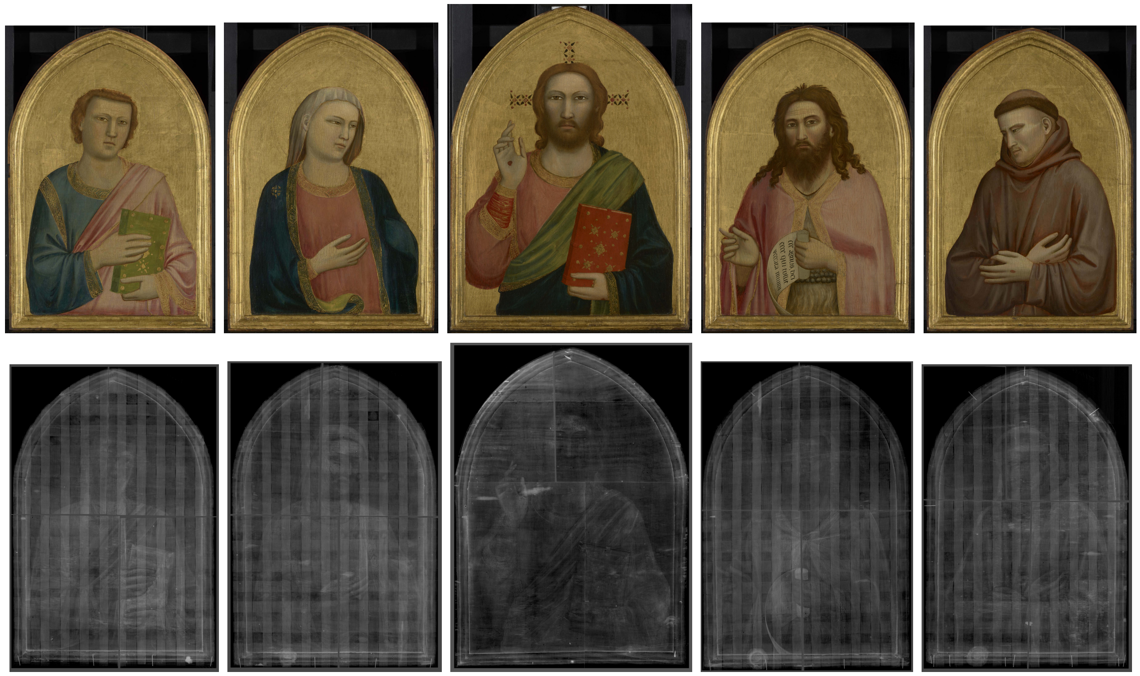

Fig. 8. Giotto di Bondone and assistants, *The Peruzzi Altarpiece*, ca. 1310-15, North Carolina Museum of Art, Raleigh. The panels from left to right are John the Evangelist, the Virgin Mary, Christ in Majesty, John the Baptist, and Francis of Assisi. The resolution of the x-ray image used in the analysis is 300 DPI. The vertical and (less obvious) horizontal stripes on the x-ray images in all panels except the central panel of Christ are caused by cradling. Each x-ray image is a mosaic of 4 x-ray films, leading to visible boundaries of the different pieces (thin horizontal and vertical lines) on the x-ray image.

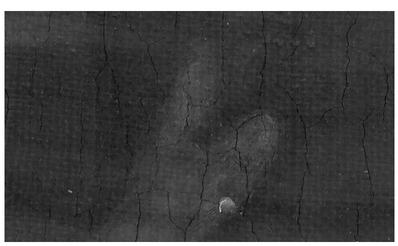

Fig. 9. A zoomed-in x-ray image of the central panel in *The Peruzzi Altarpiece*. The canvas texture is barely visible, even though the image is scaled such that the thread density is comparable with that of the zoomed-in x-ray in Figure 6.

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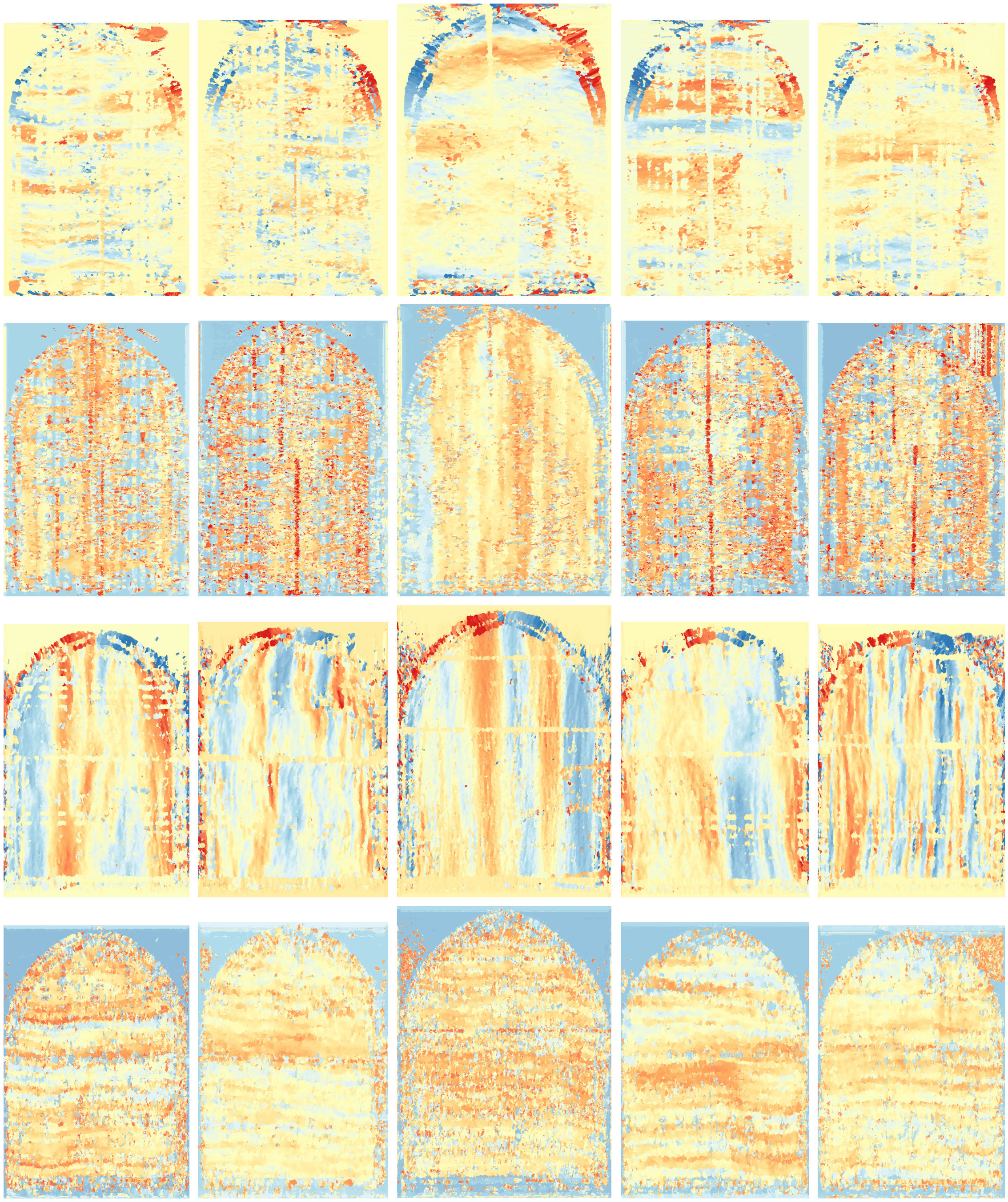

Fig. 10. Canvas analysis result of the Giotto altarpiece. First row: deviation of vertical thread angle; second row: deviation of vertical thread count; third row: deviation of horizontal thread angle; fourth row: deviation of horizontal thread count. The panels are in the same order as in Figure 8.

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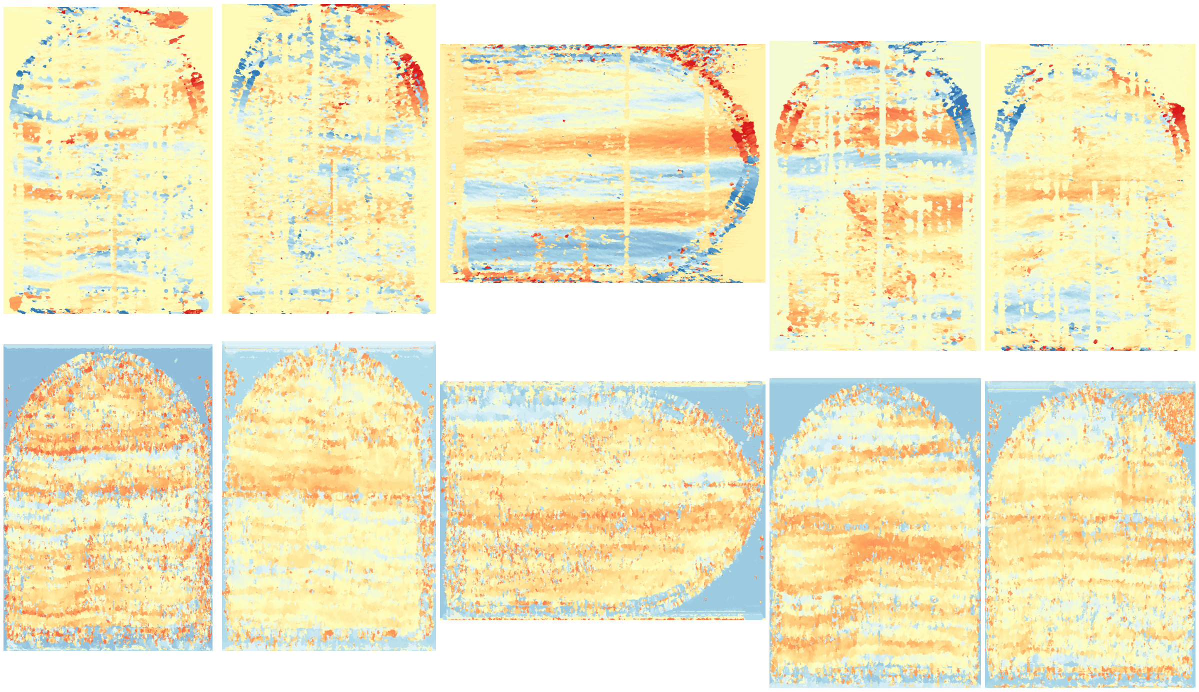

Fig. 11. A candidate canvas matching and arrangement for *The Peruzzi Altarpiece* by Giotto and his assistants. Top row: deviation of weft thread angle. Bottom row, deviation of warp thread count. (The weft thread count and warp thread angle are not shown as they are less helpful in inferring a possible arrangement.) The canvas pieces from left to right correspond to the panels for John the Evangelist, the Virgin Mary, Christ in Majesty, John the Baptist, and Francis of Assisi (in that order). The canvas of the central panel is rotated clockwise by 90 degrees, and that of the Baptist is flipped horizontally.

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