Abstract. We prove a motivic version of the Lefschetz hyperplane theorem for a smooth ample divisor $\Theta$ on an Abelian variety. We use this to construct a motive $P$ that realizes the primitive cohomology of $\Theta$.

1. Introduction

Let $k$ be an algebraically closed field. Given a smooth projective variety $X$ of dimension $d$ over $k$ and a Weil cohomology $H^*$, there is a decomposition of the diagonal $[\Delta_X] \in H^{2d}(X \times X)$ into its K"unneth components:

$$\Delta_X = \Delta_{0,X} + \ldots + \Delta_{2d,X} \in H^{2d}(X \times X) \cong \bigoplus H^j(X) \otimes H^{2d-j}(X)$$

It is one of Grothendieck’s standard conjectures ([7] Section 4) that these K"unneth components arise from algebraic cycles; i.e., that there exist correspondences $\pi_{j,X} \in CH^d(X \times X)$ for which $cl(\pi_{j,X}) = \Delta_{j,X}$ under the cycle class map $cl : CH^d(X \times X) \to H^{2d}(X \times X)$. We can state a stronger version of this conjecture as follows:

**Conjecture 1.1** (Chow-K"unneth). There exist correspondences $\pi_{j,X} \in CH^d(X \times X)$ satisfying:

(a) $\pi_{j,X}^2 = \pi_{j,X}$, $\pi_{j,X} \circ \pi_{j',X} = 0$ for $j \neq j'$

(b) $\sum \pi_{j,X} = \Delta_X$

(c) $cl(\pi_{j,X}) = \Delta_{j,X}$ for any choice of Weil cohomology.

In this stronger version, the correspondences $\pi_{j,X}$ are actually idempotents, which gives Chow motives $h^j(X) = (X, \pi_{j,X}, 0)$. Moreover, the decomposition of the diagonal into orthogonal components gives a decomposition of the motive of $X$ as $\bigoplus h^j(X)$. An important problem in the theory of motives is to understand these “underlying” objects $h^j(X)$ that represent the various degrees of cohomology (for every choice of cohomology). The Chow-K"unneth conjecture is known to hold in some important cases: curves, surfaces ([11] Chapter 6), Abelian varieties ([2]), elliptic modular varieties ([3]).

Suppose that $A$ is an Abelian variety of dimension $g$ and $i : \Theta \hookrightarrow A$ is a smooth ample divisor. The first goal of this note is then to prove the following:

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Theorem 1.1. There exist correspondences \( \pi_{j,\Theta} \in CH^{g-1}(\Theta \times \Theta) \) satisfying conjecture [1,1].

The Lefschetz hyperplane theorem gives isomorphisms \( i^* : H^j(A) \to H^j(\Theta) \) for \( j < g - 1 \) and \( i_* : H^j(\Theta) \to H^{j+2}(A) \) for \( j > g - 1 \). The proof of Theorem 1.1 gives a particular set of idempotents \( \pi_{j,\Theta} \) and we set \( h^j(\Theta) = (\Theta, \pi_{j,\Theta}, 0) \). We also set \( h^j(A) = (A, \pi_{j,A}, 0) \), where \( \pi_{j,A} \) are the canonical idempotents constructed in [2]. We are then able to prove the following motivic version of the Lefschetz hyperplane theorem:

Theorem 1.2. (a) The pull-back \( h^j(i) := \pi_{j,\Theta} \circ \Gamma_i \circ \pi_{j,A} : h^j(A) \to h^j(\Theta) \) is an isomorphism for \( j < g - 1 \).
(b) The push-forward \( h^{j+2}(i) := \pi_{j+2,A} \circ \Gamma_i \circ \pi_{j,\Theta} : h^j(\Theta) \to h^{j+2}(A)(1) \) is an isomorphism for \( j > g - 1 \).
(c) \( h^{g-1}(i) \) is split-injective and \( h^{g-1}(i) \) is split-surjective.
(d) There is an idempotent \( p \in CH^{g-1}(\Theta \times \Theta) \) which is orthogonal to \( \pi_{j,\Theta} \) for \( j \neq g - 1 \) and for which the motive \( P := (\Theta, p, 0) \) satisfies \( H^*(P) = K_{\Theta} := \ker(i_* : H^{g-1}(\Theta) \to H^{g-1}(A)) \).

We can specialize to the case that \( k = \mathbb{C} \) and \( H^* \) is singular cohomology with \( \mathbb{Q} \)-coefficients. The primitive cohomology of \( \Theta \),

\[
K_{\Theta} = \ker(i_* : H^{g-1}(\Theta, \mathbb{Q}) \to H^{g+1}(A, \mathbb{Q})(1)),
\]

is the only Hodge substructure of \( H^* \) not coming from \( A \). So, one should expect to encounter difficulty in analyzing the motive \( P \). The simplest nontrivial case is when \( A \) is a principally polarized Abelian fourfold and \([\Theta] \in CH^1(A)\) is its principal polarization. In this case, \( \Theta \) is generally a smooth divisor and \( H^*(P) = K_{\Theta} \) has Hodge level 1. Conjecturally, a motive over \( \mathbb{C} \) whose singular cohomology has Hodge level 1 should correspond to an Abelian variety ([?] Remark 7.12). We have the following partial result:

Proposition 1.1. There exists an Abelian variety \( J \) such that \( h^1(J)(-1) \cong P \iff p_*CH_0(\Theta) = 0 \).

2. Preliminaries

Let \( \mathcal{M}_k \) denote the category of Chow motives over \( k \) whose objects are triples \((X, \pi, n)\), where \( X \) is a smooth projective variety of dimension \( d \), \( \pi \in CH^d(X \times X) \) is an idempotent and \( n \in \mathbb{Z} \). The morphisms are defined as follows:

\[
\text{Hom}_{\mathcal{M}_k}((X, \pi, n), (X', \pi', n')) := \pi' \circ Cor^{n'-n}(X, X') \circ \pi
\]

\[
= \pi' \circ CH^{d+n'-n}(X \times X') \circ \pi
\]

Here, composition is defined in [3] Chapter 16.1. There is a functor \( \mathcal{Y}^{opp}_k \to \mathcal{M}_k \) from the category of smooth projective varieties over \( k \) with \( \mathcal{Y}(X) = (X, \Delta_X, 0) \) and with \( \mathcal{Y}(g) = \Gamma_g \)
for any morphism \( g : X \to X' \). A Weil cohomology theory is a functor \( H^* : Y_k^{opp} \to Vec_K \) (with \( K \) is a field of characteristic 0) satisfying certain axioms (described in [4] Section 4), one of which is the Lefschetz hyperplane isomorphism. Examples include singular, \( \ell \)-adic, crystalline, or de Rham cohomology. This extends to a functor \( H^* : \mathcal{M}_k \to Vec_K \), and for \( M = (X, \pi, m) \), we have
\[
H^j(M) = \pi_* H^{j+2m}(X).
\]
Also, there is the extension of scalars functor \( (\cdot)_L : \mathcal{M}_k \to \mathcal{M}_L \) for any field extension \( k \subset L \).

For \( M = (X, \pi, m) \), we will use the notation \( X,\pi,m \).

Lemma 2.1 (Liebermann). Let \( h_X : X' \rightrightarrows X \), \( h_Y : Y' \rightrightarrows Y \) be correspondences of smooth projective varieties. Then, for \( \alpha \in CH^*(X \times Y), \beta \in CH^*(X' \times Y') \), we have
\[(a) (h_X \times h_Y)_* (\alpha) = h_Y \circ \alpha \circ h_X \]
\[(b) \text{ When } f : X \to X' \text{ and } g : Y \to Y' \text{ are morphisms, } (f \times g)_* (\alpha) = \Gamma_g \circ \alpha \circ \Gamma_f. \]

Proof. See [3] Proposition 16.1.1. \( \square \)

Theorem 2.1 (Shermenev, Deninger-Murre). Let \( A \) be an Abelian variety of dimension \( g \) over \( k \). Then, there is a unique set of idempotents \( \{ \pi_{j,A} \} \in CH^g(A \times A) \) satisfying conjecture 1.1 and the following relation for all \( n \in \mathbb{Z} \):
\[
\Gamma_n \circ \pi_{j,A} = n^j \cdot \pi_{j,A} = \pi_{j,A} \circ \Gamma_n
\]

Proof. See [2] Theorem 3.1. \( \square \)

Let \( i : \Theta \to A \) be a smooth ample divisor and let \( \mathfrak{h}^j(A) = (A, \pi_{j,A}, 0) \) be the motive for the idempotents in Theorem 2.1. Then, we define the Lefschetz operator:
\[
L_\Theta := \Delta_* (\Theta) \in CH^{g+1}(A \times A).
\]

The most essential result for the proofs of theorems 1.1 and 1.2 is the following in [8], a motivic version of the Hard Lefschetz theorem:

Theorem 2.2 (Künnemann). Assume that \( [\Theta] = (-1)^g_\ast [\Theta] \in CH^1(A) \).

(a) \( (L_\Theta)_\ast \alpha = \alpha \cup [\Theta] \) for \( \alpha \in H^*(A) \)

(b) The operator \( \pi_{2g-j,A} \circ L_{\Theta}^{g-j} \circ \pi_{j,A} : \mathfrak{h}^j(A)(g-j) \to \mathfrak{h}^{2g-j}(A) \) is an isomorphism of motives for \( j \leq g \). That is, there exists a correspondence \( \Lambda_\Theta \in CH^{g-1}(A \times A) \) such that the following relations hold for \( j \leq g \):
\[
\pi_{j,A} \circ \Lambda_\Theta^{g-j} \circ L_{\Theta}^{g-j} \circ \pi_{j,A} = \pi_{j,A}
\]
\[
\pi_{2g-j,A} \circ L_{\Theta}^{g-j} \circ \Lambda_\Theta^{g-j} \circ \pi_{2g-j,A} = \pi_{2g-j,A}
\]

(c) Set \( \pi_{j,A} = 0 \) for all \( j \notin \{0,1,...g\} \). Then, we have \( L_{\Theta} \circ \pi_{j,A} = \pi_{j+2,A} \circ L_{\Theta} \) and \( \Lambda_\Theta \circ \pi_{j,A} = \pi_{j-2,A} \circ \Lambda_\Theta \).
Proof. See [8] Theorem 4.1. It should be noted that (b) holds more generally for Abelian schemes. It is a technical result that uses properties of the Fourier transform for Chow groups of Abelian schemes. □

By Theorem 2.2(a) and the projection formula, we have $(L_\Theta)_* = \bigcup [\Theta] = i_* \circ i^*$. The result below shows that this is true on the level of correspondences:

**Lemma 2.2.** $L_\Theta = \Gamma_i \circ \Gamma_i \in CH^{g+1}(A \times A)$.

**Proof.** From the obvious commutative diagram:

$$
\begin{array}{ccc}
\Theta & \xrightarrow{\Delta_\Theta} & \Theta \times \Theta \\
| & & | \\
i & i \times i & |
\end{array}
\xrightarrow{\Delta_A} \begin{array}{ccc}
A & \xrightarrow{\Delta_A} & A \times A
\end{array}
$$

we have $L_\Theta = (\Delta_A)_*(\Theta) = (\Delta_A)_*(i_* 1) = (i \times i)_*(\Delta_\Theta) = \Gamma_i \circ \Delta_\Theta \circ \Gamma_i = \Gamma_i \circ \Gamma_i$, where the penultimate step follows from Lemma 2.1(b). □

3. PROOFS OF THEOREMS 1.1 AND 1.2

Since $k$ is algebraically closed, it’s possible to find some $a \in A(k)$ such that $t_a^*[\Theta] \in CH^1(A)$ is invariant under $(-1)^*$. So, we can assume that $(-1)^*_A[\Theta] = [\Theta]$, so that the results of the previous section are applicable.

**Proof of Theorem 1.1.** For the proof, we will need to exhibit correspondences $\pi_{j,\Theta} \in CH^{g-1}(\Theta \times \Theta)$ which satisfy conjecture 1.1. These are given as follows:

$$
\begin{align*}
\pi_{j,\Theta} &= \tau \Gamma_i \circ \pi_{j,A} \circ [\Lambda^{g-j}_\Theta \circ L^{g-j-1}_\Theta] \circ \Gamma_i \\
&= \tau \Gamma_i \circ \pi_{j,A} \circ \Lambda^{g-j}_\Theta \circ L^{g-j-1}_\Theta \circ \Gamma_i \\
&= \tau \Gamma_i \circ \pi_{j,A} \circ \Lambda^{g-j}_\Theta \circ [\Lambda^{g-j}_\Theta \circ L^{g-j-1}_\Theta] \circ \Gamma_i \\
&= \tau \Gamma_i \circ \pi_{j,A} \circ \Lambda^{g-j}_\Theta \circ L^{g-j-1}_\Theta \circ \Gamma_i = \pi_{j,\Theta}
\end{align*}
$$

Since $\sum \pi_{j,\Theta} = \Delta_\Theta$ holds by definition, it suffices to check conditions (a) and (c) of conjecture 1.1. For $j < g - 1$, we have

$$
\begin{align*}
\pi_{j,\Theta}^2 &= \tau \Gamma_i \circ \pi_{j,A} \circ [\Lambda^{g-j}_\Theta \circ L^{g-j-1}_\Theta] \circ \Gamma_i \\
&= \tau \Gamma_i \circ \pi_{j,A} \circ [\Lambda^{g-j}_\Theta \circ L^{g-j-1}_\Theta] \circ \Gamma_i \\
&= \tau \Gamma_i \circ \pi_{j,A} \circ [\Lambda^{g-j}_\Theta \circ L^{g-j-1}_\Theta \circ \Gamma_i]
\end{align*}
$$
Here, the second equality holds by Lemma 2.2, the third holds by Theorem 2.2(b). Similarly, for \( j > g - 1 \) we have:

\[
\pi_{j,\Theta}^2 = t \Gamma_i \circ \pi_{j,A} \circ L_{\Theta}^{j-g+1} \circ \Lambda_{\Theta}^{j-g+2} \circ \pi_{j+2,A} \circ \Gamma_i = t \Gamma_i \circ L_{\Theta}^{j-g+1} \circ \Lambda_{\Theta}^{j-g+2} \circ \pi_{j+2,A} \circ \Gamma_i
\]

Similarly, the second equality holds by Lemma 2.2, and the last equality follows from the orthogonality condition.

Thus, \( \pi_{j,\Theta}^2 = \pi_{j,\Theta} \) for \( j \neq g - 1 \). Before proving the same for \( j = g - 1 \), we show that the orthogonality condition of (a) (in conjecture 1.1) holds; that is, \( \pi_{j,\Theta} \circ \pi_{j,\Theta} = 0 \) for \( j \neq j' \) and \( j, j' \neq g - 1 \). We do this for the case of \( j \neq j' < g - 1 \):

\[
\pi_{j,\Theta} \circ \pi_{j',\Theta} = t \Gamma_i \circ \pi_{j,A} \circ \Lambda_{\Theta}^{g-j} \circ L_{\Theta}^{g-j-1} \circ \Gamma_i = t \Gamma_i \circ \pi_{j',A} \circ \Lambda_{\Theta}^{g-j'} \circ L_{\Theta}^{g-j'-1} \circ \Gamma_i = 0
\]

(5)

Again, the second equality holds by Lemma 2.2 and the last equality follows from the orthogonality condition in Theorem 2.1. The third equality holds because we have

\[
\pi_{j,A} \circ \Lambda_{\Theta}^{g-j} \circ L_{\Theta}^{g-j} = \Lambda_{\Theta}^{g-j} \circ L_{\Theta}^{g-j} \circ \pi_{j,A}
\]

which follows by repeated application of Theorem 2.2(c). The remaining cases of orthogonality (\( j \neq j' \) and \( j, j' \neq g - 1 \)) are identical to (5).

What remains for the verification of condition (a) is to show that:

(i) \( \pi_{g-1,\Theta}^2 = \pi_{g-1,\Theta} \)

(ii) \( \pi_{g-1,\Theta} \circ \pi_{j,\Theta} = 0 = \pi_{j,\Theta} \circ \pi_{g-1,\Theta} \) for \( j \neq g - 1 \)

For (i) let \( \pi = \sum_{k \neq g-1} \pi_{j,\Theta} \). Since the summands are mutually orthogonal idempotents by the preceding verifications, it follows that \( \pi^2 = \pi \). Since \( \pi_{g-1,\Theta} = \Delta_{\Theta} - \pi \) by definition, we have

\[
\pi_{g-1,\Theta}^2 = (\Delta_{\Theta} - \pi)^2 = \Delta_{\Theta} + \pi^2 - 2\pi = \Delta_{\Theta} - \pi = \pi_{g-1,\Theta}
\]

For (ii) let \( j \neq g - 1 \) and note that

\[
\pi_{g-1,\Theta} \circ \pi_{j,\Theta} = (\Delta_{\Theta} - \pi) \circ \pi_{j,\Theta} = \pi_{j,\Theta} - \sum_{k \neq g-1} \pi_{k,\Theta} \circ \pi_{j,\Theta}
\]

where the third equality holds since \( \pi_{k,\Theta} \circ \pi_{j,\Theta} = 0 \) for \( j \neq k \). Similarly, one has \( 0 = \pi_{j,\Theta} \circ \pi_{g-1,\Theta} \). This completes the verification of item (a) in conjecture 1.1.

Finally, we prove (c) in conjecture 1.1. It suffices to show that \( \pi_{j,\Theta} \) acts as the identity on \( H^2(\Theta) \) and trivially on \( H^j(\Theta) \) for \( j \neq j' \) and any Weil cohomology \( H^* \). One easily reduces
Then, for \( j < g \), we need only show that \( \pi_{j,A} \) acts as the identity on \( H^j(\Theta) \). To this end, let \( \phi := \Lambda^g_{g-j} \circ L^{g-j-1}_G \circ \Gamma_i \) so that

\[
\pi_{j,\Theta} = \iota \Gamma_i \circ \pi_{j,A} \circ \phi
\]

Since \( H^* \) is a Weil cohomology, \( \iota \Gamma_i = i^* : H^j(A) \to H^j(\Theta) \) is an isomorphism (see [7]). Moreover, by Hard Lefschetz, \( (\phi \circ \iota \Gamma_i)_* = (\Lambda^g_{g-j})_* \circ (L^{g-j}_G)_* \) is the identity on \( H^j(A) \). Thus, \( i^* \) and \( \phi_* \) are inverses, from which it follows that \( (\pi_{j,\Theta})_* \) is the identity on \( H^j(\Theta) \) for \( j < g - 1 \). The case of \( j > g - 1 \) is nearly identical, only that one uses the fact that \( i_* \) is an isomorphism.

**Proof of Theorem 1.2** The statements of \( (a) \) and \( (b) \) are that \( h^j(i) \) and \( t h^j(i) \) are isomorphisms for \( j < g - 1 \) and \( j > g - 1 \), respectively. To show this, we need to construct their inverse isomorphisms:

\[
\phi_j := \pi_{j,A} \circ \Lambda^g_{g-j} \circ L^{g-j-1}_G \circ \Gamma_i \circ \pi_{j,\Theta} \text{ for } j < g - 1
\]

\[
\phi_j := \pi_{j,\Theta} \circ \iota \Gamma_i \circ L^{j-g+1}_G \circ \Lambda^{j-g+2}_{\Theta} \circ \pi_{j+2,A} \text{ for } j > g - 1
\]

Then, for \( j < g - 1 \), we have

\[
\phi_j \circ h^j(i) = \pi_{j,A} \circ \Lambda^g_{g-j} \circ L^{g-j-1}_G \circ \Gamma_i \circ \pi_{j,\Theta} \circ \iota \Gamma_i \circ \pi_{j,A}
\]

\[
= \pi_{j,A} \circ \Lambda^g_{g-j} \circ L^{g-j-1}_G \circ \Gamma_i \circ \pi_{j,A} \circ \Lambda^g_{g-j} \circ L^{g-j-1}_G \circ \Gamma_i \circ \pi_{j,A}
\]

\[
= \pi_{j,A} \circ \Lambda^g_{g-j} \circ L^{g-j}_G \circ \pi_{j,A} \circ \Lambda^g_{g-j} \circ L^{g-j}_G \circ \pi_{j,A}
\]

\[
= \pi_{j,A}
\]

where the third and fourth equalities hold by Theorem 2.2(2). Similarly, we have

\[
h^j(i) \circ \phi_j = \pi_{j,\Theta} \circ \iota \Gamma_i \circ \pi_{j,A} \circ \Lambda^g_{g-j} \circ L^{g-j-1}_G \circ \Gamma_i \circ \pi_{j,\Theta}
\]

\[
= \pi_{j,\Theta}^3 = \pi_{j,\Theta}
\]

We conclude that \( h^j(i) \) and \( \phi_j \) are inverses for \( j < g - 1 \), proving \( (a) \). For \( (b) \), we have

\[
t h^j(i) \circ \phi_j = \pi_{j+2,A} \circ \Gamma_i \circ \pi_{j,\Theta} \circ \iota \Gamma_i \circ L^{j-g+1}_G \circ \Lambda^{j-g+2}_{\Theta} \circ \pi_{j+2,A}
\]

\[
= \pi_{j+2,A} \circ \Gamma_i \circ L^{j-g+1}_G \circ \Lambda^{j-g+2}_{\Theta} \circ \pi_{j+2,A} \circ \Gamma_i \circ \iota \Gamma_i \circ L^{j-g+1}_G \circ \Lambda^{j-g+2}_{\Theta} \circ \pi_{j+2,A}
\]

\[
= \pi_{j+2,A} \circ L^{j-g+2}_G \circ \Lambda^{j-g+2}_{\Theta} \circ \pi_{j+2,A} \circ L^{j-g+2}_G \circ \Lambda^{j-g+2}_{\Theta} \circ \pi_{j+2,A}
\]

\[
= \pi_{j,A}
\]

Similarly, we have

\[
\phi_j \circ t h^j(i) = \pi_{j,\Theta} \circ \iota \Gamma_i \circ L^{j-g+1}_G \circ \Lambda^{j-g+2}_{\Theta} \circ \pi_{j+2,A} \circ \Gamma_i \circ \pi_{j,\Theta}
\]

\[
= \pi_{j,\Theta}^3 = \pi_{j,\Theta}
\]
So, \(^\dagger\)\(\mathfrak{h}^j(i)\) and \(\phi_j\) are inverses for \(j > g - 1\). For \([c]\) we need to show that \(\mathfrak{h}^{g-1}(i)\) and \(\dagger\mathfrak{h}^{g-1}(i)\) are split-injective and split-surjective, respectively. Their left and right inverses will be:

\[
\phi_{g-1} = \pi_{g-1,A} \circ \Lambda_\Theta \circ \Gamma_i \circ \pi_{g-1,\Theta} \\
\psi_{g-1} = \pi_{g-1,\Theta} \circ \dagger \Gamma_i \circ \Lambda_\Theta \circ \pi_{g+1,A}.
\]

(6)

To this end, we begin by noting that for \(j < g - 1\):

\[
\pi_{j,\Theta} \circ \dagger \Gamma_i = \dagger \Gamma_i \circ \pi_{j,A} \circ \Lambda_{\Theta}^{j-g} \circ L_{\Theta}^{g-j}
\]

(7)

Similarly, we have \(\Gamma_i \circ \pi_{j,\Theta} = \pi_{j+2,A} \circ \Gamma_i\) for \(j > g - 1\). So, we write \(\pi = \sum_{j \neq g-1} \pi_{j,\Theta}\) as before and obtain:

\[
\Gamma_i \circ \pi \circ \dagger \Gamma_i \circ \pi_{g-1,A} = \sum_{j < g-1} \Gamma_i \circ \pi_{j,\Theta} \circ \dagger \Gamma_i \circ \pi_{g-1,A} + \sum_{j > g-1} \Gamma_i \circ \pi_{j,\Theta} \circ \dagger \Gamma_i \circ \pi_{g-1,A} \\
= \sum_{j < g-1} \Gamma_i \circ \dagger \Gamma_i \circ \pi_{j,A} \circ \pi_{g-1,A} + \sum_{j > g-1} \pi_{j+2,A} \circ \Gamma_i \circ \dagger \Gamma_i \circ \pi_{g-1,A}
\]

(8)

\[
= \sum_{j < g-1} L_{\Theta} \circ \pi_{j,A} \circ \pi_{g-1,A} + \sum_{j > g-1} \pi_{j+2,A} \circ L_{\Theta} \circ \pi_{g-1,A}
\]

\[
= \sum_{j > g-1} L_{\Theta} \circ \pi_{j,A} \circ \pi_{g-1,A} = 0
\]

where the third equality holds by the mutual orthogonality of \(\pi_{j,A}\) and the fourth holds because \(L_{\Theta} \circ \pi_{j,A} = \pi_{j+2,A} \circ L_{\Theta}\). Thus, we have:

\[
\phi_{g-1} \circ \mathfrak{h}^{g-1}(i) = \pi_{g-1,A} \circ \Lambda_\Theta \circ \Gamma_i \circ \pi_{g-1,\Theta} \circ \dagger \Gamma_i \circ \pi_{g-1,A} \\
= \pi_{g-1,A} \circ \Lambda_\Theta \circ \Gamma_i \circ (\Delta_\Theta - \pi) \circ \dagger \Gamma_i \circ \pi_{g-1,A} \\
= \pi_{g-1,A} \circ \Lambda_\Theta \circ \Gamma_i \circ \dagger \Gamma_i \circ \pi_{g-1,A} - \pi_{g-1,A} \circ \Lambda_\Theta \circ \Gamma_i \circ \pi \circ \dagger \Gamma_i \circ \pi_{g-1,A} \\
= \pi_{g-1,A} \circ \Lambda_\Theta \circ L_{\Theta} \circ \pi_{g-1,A} = \pi_{g-1,A}
\]

Here, the second term on the third line vanishes by \([8]\). So, \(\mathfrak{h}^{g-1}(i)\) is split-injective. A similar calculation shows that \(\dagger \mathfrak{h}^{g-1}(i)\) is split-surjective with right inverse \(\psi_j\). The completes the proof of \([c]\)

Finally, for \([d]\) we define:

\[
\pi_{g-1,\Theta} := \dagger \Gamma_i \circ \pi_{g-1,A} \circ \Lambda_\Theta \circ \Gamma_i \in CH^{g-1}(\Theta \times \Theta)
\]

As in the proof of Theorem \([1.1]\) one can show that \(\pi_{g-1,\Theta}\) is an idempotent, is orthogonal to \(\pi_{j,\Theta}\) for \(j \neq g - 1\). It follows that

\[
\pi_{g-1,\Theta} \circ \pi_{g-1,\Theta} = \pi_{g-1,\Theta} - \sum_{j \neq g-1} \pi_{g-1,\Theta} \circ \pi_{j,\Theta} = \pi_{g-1,\Theta}
\]
Similarly, one has $\pi_{g-1,\Theta} \circ \pi'_{g-1,\Theta} = \pi'_{g-1,\Theta}$. Write $h^{-1}_q(\Theta) = (\Theta, \pi_{g-1,\Theta}, 0)$ for the corresponding motive and define:

$$p := \pi_{g-1,\Theta} - \pi'_{g-1,\Theta} \in CH^{g-1}(\Theta \times \Theta)$$

We have

$$p^2 = (\pi_{g-1,\Theta} - \pi'_{g-1,\Theta})^2 = \pi_{g-1,\Theta}^2 + (\pi'_{g-1,\Theta})^2 - 2\pi_{g-1,\Theta} \circ \pi'_{g-1,\Theta}$$

$$= \pi_{g-1,\Theta} + \pi'_{g-1,\Theta} - 2\pi'_{g-1,\Theta} \circ \pi'_{g-1,\Theta} = \pi_{g-1,\Theta} - \pi'_{g-1,\Theta} = p$$

so that $p$ is an idempotent. Write $P := (\Theta, p, 0)$ for the corresponding motive. We also have

$$p \circ \pi'_{g-1,\Theta} = (\pi_{g-1,\Theta} - \pi'_{g-1,\Theta}) \circ \pi'_{g-1,\Theta} = \pi'_{g-1,\Theta} - \pi'_{g-1,\Theta} = 0$$

so that $p$ and $\pi'_{g-1,\Theta}$ are orthogonal. This gives a decomposition of motives:

$$h^{-1}(\Theta) = P \oplus h^{-1}_q(\Theta)$$

The same argument for Theorem 1.1 (c) shows that $H^*(h^{-1}_q(\Theta)) = i^*H^{g-1}(\Theta)$. Thus, applying $H^*$ to (9), it follows that $H^*(P) = K_{\Theta}$. 

4. The complementary motive $P$

Now, let $k = \mathbb{C}$ and $H^*$ be singular cohomology with $\mathbb{Q}$-coefficients. We consider the case of $A$ a principally polarized Abelian variety, whose principal polarization is the class of $i: \Theta \to A$. Since we are interested in the motive $P$, we need $\Theta$ to be nonsingular. The simplest nontrivial case is that of $g = 4$, where a well-known result of Mumford in [9] is that $\Theta$ is generally nonsingular. Now, let $K_{\Theta, \mathbb{Q}} := \ker(i_*: H^{g-1}(\Theta, \mathbb{Q}) \to H^{g+1}(A, \mathbb{Q}(1)))$ be the primitive cohomology. Then, we have the following:

**Lemma 4.1.** $K_{\Theta}$ is a rational Hodge structure of level 1 and dimension 10.

**Proof.** Since $H^3(\Theta)$ and $H^3(A)$ both have Hodge level 3, we need to show that $i^* : H^{3,0}(A) \to H^{3,0}(\Theta)$ is an isomorphism. Since this map is already injective, it will suffice to show that $h^{3,0}(\Theta) = h^{3,0}(A) = 4$. By adjunction, $\omega_{\Theta} \cong \mathcal{O}_\Theta(\Theta)$, so $h^0(\Theta, \mathcal{O}_\Theta(\Theta)) = h^{3,0}(\Theta)$. We can use the long exact sequence to compute $h^0(\Theta, \mathcal{O}_\Theta(\Theta))$:

$$0 \to H^0(A, \mathcal{O}_A) \to H^0(A, \mathcal{O}_A(\Theta)) \to H^0(\Theta, \mathcal{O}_\Theta(\Theta)) \to H^1(A, \mathcal{O}_A) \to H^1(A, \mathcal{O}_A(\Theta)) = 0$$

Since $\Theta$ is a principal polarization, $h^0(A, \mathcal{O}_A(\Theta)) = 1$ so that the restriction arrow is 0.

Moreover, $h^1(A, \mathcal{O}_A) = 4$, so it follows that $h^{3,0}(\Theta) = 4 = h^{3,0}(A)$. Thus, $i^* : H^{3,0}(A) \to H^{3,0}(\Theta)$ is an isomorphism and $K_{\Theta}$ has Hodge level 1. To determine the dimension of $K_{\Theta}$, we first compute $\chi(\Theta) = c_3(T\Theta)$. Applying the Chern polynomial to the adjunction sequence in this case, one obtains that $c_3(T\Theta) = -c_1(\mathcal{O}(\Theta))^4 = -4! = -24$. Using the Lefschetz hyperplane theorem, one also computes that $\chi(\Theta) = 42 - h^3(\Theta)$, so that $h^3(\Theta) = 66$. Since $h^3(A) = \binom{8}{3} = 56$, it follows that $K_{\Theta}$ has dimension 10. 

□
Thus, $H^*(P, \mathbb{Q})$ has Hodge level 1 when $g = 4$. Now, consider the intermediate Jacobian $J(K_\Theta)$:

$$J(K_\Theta) = K_{\Theta, \mathbb{C}}/(F^2K_{\Theta, \mathbb{C}} \oplus K_{\Theta, \mathbb{Z}})$$

This is a principally polarized Abelian variety of dimension 5, and we have an isomorphism of rational Hodge structures $H^1(J(K_\Theta), \mathbb{Q})(-1) \cong H^3(P, \mathbb{Q})$. The generalized Hodge conjecture predicts that this isomorphism arises from a correspondence $\Gamma \subset J(K_\Theta) \times \Theta$. The existence of $\Gamma$ was proved in [5]. One may take this a step further and ask whether $h_1(J(K_\Theta))(-1)$ and $P$ are isomorphic as motives. Proposition 1.1 provides a partial answer to this; i.e., we have $h_1(J(K_\Theta))(-1) \cong P$ if $p$ acts trivially on $CH_0(\Theta_L)$ for all field extensions $\mathbb{C} \subset L$ (and conversely). We will need the following definition for the proof:

**Definition 4.1.** We say that $M = (X, \pi, 0) \in M_k$ has representable Chow group in codimension $i$ if there exists a smooth complete (possible reducible) curve $C$ and $\Gamma \in CH^i(C \times X)$ such that $CH^i_{alg}(M_L) = \pi_L^*CH^i_{alg}(X_L)$ lies in $\Gamma_LCH^1_{alg}(C_L) \subset CH^1_{alg}(X_L)$ for every field extension $k \subset L$.

**Proof of Proposition 1.1.** Suppose that we have some Abelian variety $J$ for which $h_1(J)(-1) \cong P$. Then, applying $CH^3(\ )$ to both sides we obtain

$$p_\ast CH_0(\Theta) = p_\ast CH^3(\Theta) \cong CH^3(h_1(J)(-1)) = CH^2(h_1(J))$$

From [2] Theorem 2.19, we have $CH^2(h_1(J)) = 0$ so that $p_\ast CH_0(\Theta) = 0$. For the converse, observe that $\Theta$ can be defined over some field $k$ which is the algebraic closure of a finitely generated over $\mathbb{Q}$. So, let $\Theta_k$ be a model for $\Theta$ over $k$. The operators used in the proof of Theorems 1.1 and 1.2 ($L_\Theta$, $A_\Theta$, and $\pi_{j,A}$) are well-behaved upon passage to an overfield (see [2] and [8]); thus, so is the correspondence $p_k \in Cor^0(\Theta_k \times \Theta_k)$ constructed above. This means that $p_C$ coincides with $p$ (as in the statement of Proposition 1.1), and the assumption that $p$ acts trivially on $CH_0$ becomes the assumption that

$$p_L_\ast CH^3(\Theta_L) = 0$$

for all overfields $k \subset L$. Now, let $P = (\Theta_k, p_k, 0)$. The task is then to find an Abelian variety $J$ over $k$ for which

$$h_1(J)(-1) \cong P$$

To this end, we begin with the following lemma:

**Lemma 4.2.** $P$ has representable Chow group in codimension 2.

**Proof of Lemma.** We will drop the subscript $k$. We use the same argument as in [1]. There is a localization sequence:

$$\lim_{D \subset \Theta} CH^2(\Theta \times D) \xrightarrow{(\text{id}_D \times \text{id}_J)^\ast} CH^3(\Theta \times \Theta) \xrightarrow{(\text{id}_D \times K)^\ast} CH^3(\Theta_K) \longrightarrow 0$$
where the limit runs over all (possibly reducible) subvarieties $D$ of codimension 1 and $K = \mathbb{C}(\Theta)$ is the function field of $\Theta$. We have $(id_\Theta \times K)^* \Delta_\Theta = \eta_K$, the generic point of $\Theta$. From Lemma 2.2(a), we have $p = p \circ \Delta_\Theta = (p \times id_\Theta)_\ast \Delta_\Theta$ so that

$$(id_\Theta \times K)^*(p) = (id_\Theta \times K)^*(p \times id_\Theta)_\ast \Delta_\Theta = p_K \ast (id_\Theta \times K)^* \Delta_\Theta = p_K \ast (\eta_K)$$

Since $p_K \ast (\eta_K) = 0$ by assumption, the exactness of (10) gives some subvariety $D$ and $\alpha \in \text{CH}^2(\Theta \times D)$ for which $p = (id_\Theta \times j_D) \ast \alpha$. After desingularizing, we can assume that $D$ is smooth (although $j_D$ may no longer be an inclusion). By Lemma 2.2(b), we have

$$p = (id_\Theta \times j_D) \ast \alpha \cdot \Gamma_{j_D} \circ \alpha$$

Thus $p_{L^s} CH^2_{alg}(\Theta_L) \subset j_{D_L^s} CH^1_{alg}(D_L)$. By the representability of the Picard functor, this means there is some smooth complete $C$ and some $\Gamma \in CH^1(C \times D)$ such that $\Gamma_{L^s} CH^1_{alg}(C_L) = CH^1_{alg}(D_L)$ for all field extension $k \subset L$. This proves the lemma. \hfill \Box

Thus, we see that the Chow group of $P$ is representable in every codimension. By [12] Theorem 3.4, it follows that the motive of $P$ decomposes as

$$\bigoplus I(i)^{\oplus n_i} \oplus h^1(J_i)(-i)$$

for integers $n_i$ and Abelian varieties $J_i$. Since the cohomology of $P$ is 0 in all but degree 3, this means that $P \cong h^1(J)(-1)$ for some Abelian variety $J$. This gives the proposition. \hfill \Box

**Remark 4.1.** A more refined version of Proposition [12] is that the Abelian variety can be taken to be $J(K_\Theta)$ in the above notation. Indeed, since we have $H^1(J(K_\Theta), \mathbb{Q})(-1) \cong H^3(P, \mathbb{Q}) \cong H^1(J, \mathbb{Q})(-1)$ (as rational Hodge structures), it follows that $J$ and $J(K_\Theta)$ are isogenous.

**References**


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