Pythagorean Triples with a Fixed Difference between a Leg and the Hypotenuse

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Abstract

This project examines sets of Pythagorean triples with a fixed difference k between one of the legs and the hypotenuse. A Pythagorean triple is an ordered triple of positive integers (x, y, z) such that $x^2 + y^2 = z^2$. This paper proposes and proves a formula that will generate every triple of a difference k, that is, every triple where either $z - x = k$ or $z - y = k$. The proof of this result utilizes the general formula for all primitive Pythagorean triples as well as the prime factorization of the integer k.

1 Introduction

Pythagorean triples are a very ancient area in mathematics and throughout the centuries numerous mathematicians have deepened our understanding of their characteristics. Sierpinski provides an excellent summary on some of their most well known properties in [6]. By far the most significant discovery in this area was made by Euclid (c. 350 B.C.), who found a formula that characterized every primitive Pythagorean triple [2]. One of the first known references to Pythagorean triples of a fixed difference is found in the writings of Plato. Plato notes that $(4n, 4n^2 - 1, 4n^2 + 1)$ yields an infinite set of Pythagorean triples of difference 2 as n ranges over the positive integers [1]. Building on these foundations, this project sought to find a formula that would yield every Pythagorean triple of an arbitrary positive difference k. Numerous formulas for specific k values were found and analyzed until a correct formula was obtained. As noted, the proof of the formula utilizes Euclid's classification of primitive Pythagorean triples and well-known number theoretic properties. Although similar results have been published [4], the formula and its proof were arrived at independently.

2 Formula for Pythagorean Triples of a Difference k

We have the following theorem as our main result:

Theorem 1. *Define*

$$f(k) = 2^{\lfloor \frac{n_0}{2} \rfloor} p_1^{\lfloor \frac{n_1}{2} \rfloor} p_2^{\lfloor \frac{n_2}{2} \rfloor} \ldots p_r^{\lfloor \frac{n_r}{2} \rfloor}, \tag{2.1}$$

where $2^{n_0} p_1^{n_1} p_2^{n_2} \ldots p_r^{n_r}$ is the prime factorization of $2k$. Then every triple of difference k is given by:

$$\left(\frac{k}{f(k)}(2m + k \bmod 2), \quad \frac{2k}{f(k)^2}\left(m - \left\lfloor \frac{f(k)}{2} \right\rfloor\right)\left(m + \left\lceil \frac{f(k)}{2} \right\rceil\right), \quad \frac{2k}{f(k)^2}\left(m - \left\lfloor \frac{f(k)}{2} \right\rfloor\right)\left(m + \left\lceil \frac{f(k)}{2} \right\rceil\right) + k \right)$$

as m runs through all positive integers greater than $\lfloor \frac{f(k)}{2} \rfloor$.

Now by Euclid's general formula for all primitive Pythagorean triples [2], any triple can be written as $(l(2mn), l(m^2 - n^2), l(m^2 + n^2))$, where l is some positive integer and $m > n > 0$ are two relatively prime positive integers of opposite parity. Throughout this paper x will be used to designate the leg that is a multiple of $2mn$, y will be used to designate the leg that is a multiple of $m^2 - n^2$, and z will designate the hypotenuse. Thus (x, y, z) and (y, x, z) are simply two different notations for the same triple and will be used interchangeably.

We will discuss briefly the outline of the proof of this theorem, and then we will proceed to the actual proof. Let S_k be the set of all triples of difference k, that is, the set of every triple where the hypotenuse is separated by k from a leg. Let G_k be the set of all triples generated by the above formula as m runs over all the positive integers greater than $\lfloor \frac{f(k)}{2} \rfloor$. We will show that these sets are equal, that is, that $S_k \subseteq G_k$ and that $G_k \subseteq S_k$. To show that $S_k \subseteq G_k$, we will choose an arbitrary triple (x, y, z) of difference k and show that it is infact generated by the formula given in Theorem 1. In doing this we will examine two cases: k is even and k is odd. When k is even, we will have two subcases: $z - x = k$ and $z - y = k$. When k is odd, however, we will have only $z - x = k$. After showing that in every case the triple (x, y, z) is generated by the formula given in Theorem 1, we conclude that $(x, y, z) \in G_k$ and thus that $S_k \subseteq G_k$. To show that $G_k \subseteq S_k$, we pick an arbitrary element (q, r, s) that is generated by the formula given in Theorem 1, and we show that it is infact a Pythagorean triple of difference k. To do this, we have to verify that q, r, and s are all positive integers, that $q^2 + r^2 = s^2$, and that either $s - q = k$ or $s - r = k$. After showing that (q, r, s) is indeed a Pythagorean triple of difference k, we conlcude that $(q, r, s) \in S_k$ and therefore that $G_k \subseteq S_k$. We will then have shown that the sets are equal and thus that the formula given in Theorem 1 produces every triple of differnce k and nothing else. We will now begin the actual proof.

First of all, we show that $S_k \subseteq G_k$. Let k be an arbitrary positive integer and let $(x, y, z) \in S_k$. We will look at the cases where k is even and where k is odd, and we will show that in both cases $(x, y, z) \in G_k$.

Case 1. Let k be an even integer. By the general formula for primitive Pythagorean triples, (x, y, z) is of the form:

$$\begin{aligned} x &= l(2mn) \\ y &= l(m^2 - n^2) \\ z &= l(m^2 + n^2), \end{aligned} \tag{2.2}$$

where l is some positive integer and $m > n > 0$ are two relatively prime positive integers of opposite parity. Now since this triple has a difference k, either $z - x = k$ or $z - y = k$. We will examine both of these two subcases.

Subcase 1: Suppose that $z - y = k$. Then $l(m^2 + n^2) - l(m^2 - n^2) = k$, which yields $k = 2ln^2$. Thus $n = \sqrt{\frac{k}{2l}}$, and substituting this expression for n into equation 2.2, we get:

$$\begin{aligned} x &= m\sqrt{2lk} \\ y &= lm^2 - \tfrac{k}{2} \\ z &= lm^2 + \tfrac{k}{2}. \end{aligned} \tag{2.3}$$

We will use the following lemma to show $(x, y, z) \in G_k$:

Lemma 1. *If (x, y, z) is a triple where $z - y = k$, and thus has the form given above, it can be written in the form*

$$\begin{aligned} x &= m_1\sqrt{2l_1k} \\ y &= l_1m_1^2 - \tfrac{k}{2} \\ z &= l_1m_1^2 + \tfrac{k}{2}, \end{aligned} \tag{2.4}$$

for some integers l_1 and m_1, where l_1 is the smallest positive integer such that $2l_1k$ is a perfect square and $m_1 = md$, where $d = \frac{\sqrt{2lk}}{\sqrt{2l_1k}}$.

Proof. Let $q = \sqrt{2l_1k}$, where l_1 is the smallest positive integer such that $2l_1k$ is a perfect square. Let $p = \sqrt{2lk}$, which must be an integer since x is an integer. We will show that q divides p. Now $q = \sqrt{2l_1k} = \sqrt{2^{n_0}p_1^{n_1}...p_r^{n_r}l_1}$, where $2^{n_0}p_1^{n_1}...p_r^{n_r}$ is the prime factorization of $2k$. Now let $p_{o1}, p_{o2}, ..., p_{oj}$ be the primes with odd exponents and let $p_{e1}, p_{e2}, ..., p_{e_s}$ be the primes with even exponents. Since l_1 is the smallest integer such that $q \in Z$, $l_1 = p_{o1}p_{o2}...p_{oj}$. Now let $n_{o1}, n_{o2}, ..., n_{oj}$ be the odd exponents and let $n_{e1}, n_{e2}, ..., n_{es}$ be the even exponents. Then

$$q = \sqrt{{p_{e1}}^{n_{e1}}{p_{e2}}^{n_{e2}}...{p_{e_s}}^{n_{es}}{p_{o1}}^{n_{o1}+1}{p_{o2}}^{n_{o2}+1}...{p_{oj}}^{n_{oj}+1}}. \tag{2.5}$$

Now consider $p = \sqrt{2lk}$. We have:

$$p = \sqrt{{p_{e1}}^{n_{e1}}{p_{e2}}^{n_{e2}}...{p_{e_s}}^{n_{es}}{p_{o1}}^{n_{o1}}{p_{o2}}^{n_{o2}}...{p_{oj}}^{n_{oj}}l}. \tag{2.6}$$

Thus in order for p to be an integer, l must contain at least one copy of each of the primes with odd exponents. Thus $p_{o1}p_{o2}...p_{oj}|l$. Let $r = \frac{l}{p_{o1}p_{o2}...p_{oj}}$. Thus:

$$p = \sqrt{{p_{e1}}^{n_{e1}}{p_{e2}}^{n_{e2}}...{p_{e_s}}^{n_{es}}{p_{o1}}^{n_{o1}+1}{p_{o2}}^{n_{o2}+1}...{p_{oj}}^{n_{oj}+1}r}. \tag{2.7}$$

Now since every prime exponent is even, every exponent in the prime factorization of r must also be even, or p won't be an integer. Thus r must be a perfect square, and $\sqrt{r} = d$, for some positive integer d. Thus:

$$\begin{aligned}\frac{p}{q} &= \frac{\sqrt{{p_{e1}}^{ne1}{p_{e2}}^{ne2}...{p_{e_s}}^{nes}{p_{o1}}^{no1+1}{p_{o2}}^{no2+1}...{p_{oj}}^{noj+1}r}}{\sqrt{{p_{e1}}^{ne1}{p_{e2}}^{ne2}...{p_{e_s}}^{nes}{p_{o1}}^{no1+1}{p_{o2}}^{no2+1}...{p_{oj}}^{noj+1}}} \\ &= \sqrt{r} \ = d.\end{aligned}$$

Thus $p = qd$, and $q|p$. Now let $m_1 = md$. Then:

$$\begin{aligned}&m_1\sqrt{2l_1k} = md\sqrt{2l_1k} = m\sqrt{2l_1kr} = m\sqrt{2lk} = x \\ &(\text{since } l = p_{o1}p_{o2}...p_{oj}r = l_1r) \\ &l_1m_1^2 - \tfrac{k}{2} = l_1(md)^2 - \tfrac{k}{2} = l_1m^2r - \tfrac{k}{2} = lm^2 - \tfrac{k}{2} = y \\ &l_1m_1^2 + \tfrac{k}{2} = l_1(md)^2 + \tfrac{k}{2} = l_1m^2r + \tfrac{k}{2} = lm^2 + \tfrac{k}{2} = z.\end{aligned} \tag{2.8}$$

□

Thus we have shown that the triple (x, y, z) we selected, where $z - y = k$, can be written in the form:

$$\begin{aligned}x &= m_1\sqrt{2l_1k} \\ y &= l_1m_1^2 - \tfrac{k}{2} \\ z &= l_1m_1^2 + \tfrac{k}{2},\end{aligned} \tag{2.9}$$

for some integers l_1 and m_1, where l_1 is the smallest positive integer such that $2l_1k$ is a perfect square. Now we need to show that this triple is in the set G_k. We will do this by showing that there exists some integer $m_2 > \lfloor\frac{f(k)}{2}\rfloor$ such that:

$$\begin{aligned}x &= \tfrac{k}{f(k)}(2m_2 + k \bmod 2) \\ y &= \tfrac{2k}{f(k)^2}(m_2 - \lfloor\tfrac{f(k)}{2}\rfloor)(m_2 + \lceil\tfrac{f(k)}{2}\rceil) \\ z &= \tfrac{2k}{f(k)^2}(m_2 - \lfloor\tfrac{f(k)}{2}\rfloor)(m_2 + \lceil\tfrac{f(k)}{2}\rceil) + k.\end{aligned} \tag{2.10}$$

Before we can show this, however, we must prove an additional lemma, which holds regardless of the parity of k.

Lemma 2. *If l_1 is the smallest positive integer such that $2l_1k$ is a perfect square and $f(k)$ is as defined in Theorem 1, then $l_1 = \frac{2k}{f(k)^2}$.*

Proof. Let $2^{n_0}p_1^{n_1}p_2^{n_2}...p_r^{n_r}$ be the prime factorization of $2k$. Now let $p_{o1}, p_{o2}, ..., p_{oj}$ be the primes with odd exponents and let $p_{e1}, p_{e2}, ..., p_{es}$ be the primes with even exponents; also, let $n_{e1}, n_{e2}, ..., n_{es}$ be the even exponents and let $n_{o1}, n_{o2}, ..., n_{oj}$ be the odd exponents. Then:

$$\frac{2k}{f(k)^2} = \frac{2^{n_0}p_1^{n_1}p_2^{n_2}...p_r^{n_r}}{(2^{\lfloor\frac{n_0}{2}\rfloor}p_1^{\lfloor\frac{n_1}{2}\rfloor}p_2^{\lfloor\frac{n_2}{2}\rfloor}...p_r^{\lfloor\frac{n_r}{2}\rfloor})^2}$$

$$= \frac{p_{e1}{}^{n_{e1}}p_{e2}{}^{n_{e2}}...p_{es}{}^{n_{es}}p_{o1}{}^{n_{o1}}p_{o2}{}^{n_{o2}}...p_{oj}{}^{n_{oj}}}{(p_{e1}{}^{\lfloor\frac{n_{e1}}{2}\rfloor}p_{e2}{}^{\lfloor\frac{n_{e2}}{2}\rfloor}...p_{es}{}^{\lfloor\frac{n_{es}}{2}\rfloor})^2(p_{o1}{}^{\lfloor\frac{n_{o1}}{2}\rfloor}p_{o2}{}^{\lfloor\frac{n_{o2}}{2}\rfloor}...p_{oj}{}^{\lfloor\frac{n_{oj}}{2}\rfloor})^2}$$

$$= \frac{p_{e1}{}^{n_{e1}}p_{e2}{}^{n_{e2}}...p_{es}{}^{n_{es}}p_{o1}{}^{n_{o1}}p_{o2}{}^{n_{o2}}...p_{oj}{}^{n_{oj}}}{p_{e1}^{n_{e1}}p_{e2}^{n_{e2}}...p_{es}^{n_{es}}p_{o1}^{n_{o1}-1}p_{o2}^{n_{o2}-1}...p_{oj}^{n_{oj}-1}}$$

$$= p_{o1}p_{o2}...p_{oj} = l_1.$$

(Since l_1 is the smallest integer such that $2l_1k$ is a perfect square, it contains exactly one copy of each prime that has an odd exponent in the prime factorization of $2k$.) ☐

Now we will proceed to show that the integer m_1, as described in Lemma 1, is the integer that will generate the triple (x, y, z) when inserted into our formula for G_k, as given in Theorem 1. Since $y > 0$ and $y = l_1m_1^2 - \frac{k}{2}$, we have:

$$l_1m_1^2 - \tfrac{k}{2} > 0$$

$$l_1m_1^2 > \tfrac{k}{2}$$

$$\tfrac{2k}{f(k)^2}m_1^2 > \tfrac{k}{2} \qquad \text{(using the substitution from Lemma 2)}$$

$$4m_1^2 > f(k)^2 \tag{2.11}$$

$$2m_1 > f(k)$$

$$m_1 > \tfrac{f(k)}{2}$$

$$m_1 > \lfloor\tfrac{f(k)}{2}\rfloor.$$

Thus since m_1 is an integer and $m_1 > \lfloor\frac{f(k)}{2}\rfloor$, m_1 is in the domain of the formula given in Theorem 1. Now since k is even, we have:

$$\begin{aligned}
x &= m_1\sqrt{2l_1k} \\
&= m_1\sqrt{\tfrac{2k2k}{f(k)^2}} \\
&= 2m_1\tfrac{k}{f(k)} \\
&= \tfrac{k}{f(k)}(2m_1 + k \bmod 2) \\
\\
y &= l_1m_1^2 - \tfrac{k}{2} \\
&= \tfrac{2k}{f(k)^2}m_1^2 - \tfrac{2k}{4} \\
&= \tfrac{2k}{f(k)^2}(m_1^2 - \tfrac{f(k)^2}{4}) \\
&= \tfrac{2k}{f(k)^2}(m_1 - \tfrac{f(k)}{2})(m_1 + \tfrac{f(k)}{2}) \\
&= \tfrac{2k}{f(k)^2}(m_1 - \lfloor\tfrac{f(k)}{2}\rfloor)(m_1 + \lceil\tfrac{f(k)}{2}\rceil) \\
\\
z &= l_1m_1^2 + \tfrac{k}{2} \\
&= l_1m_1^2 - \tfrac{k}{2} + k \\
&= \tfrac{2k}{f(k)^2}(m_1 - \lfloor\tfrac{f(k)}{2}\rfloor)(m_1 + \lceil\tfrac{f(k)}{2}\rceil) + k.
\end{aligned} \tag{2.12}$$

Note that since k is even, $f(k)$ is also clearly even, since $2k$ has at least two factors of 2 in it. Thus $\frac{f(k)}{2} = \lfloor\frac{f(k)}{2}\rfloor = \lceil\frac{f(k)}{2}\rceil$, which was used in the above derivation. Thus we have shown that when k is even and when $z - y = k$, $(x, y, z) \in G_k$ because the integer m_1 produces it.

Subcase 2: Now suppose that $z - x = k$ (k still even). We will make use of the following lemma to show that $(x, y, z) \in G_k$. Note, however, that this lemma does not require k to be even.

Lemma 3. *If (x, y, z) is a triple such that $z - x = k$, then there exist integers m_2 and l_2 such that*

$$\begin{aligned}
x &= 2m_2(m_2l_2 - \sqrt{kl_2}) \\
y &= 2m_2\sqrt{kl_2} - k \\
z &= 2m_2(m_2l_2 - \sqrt{kl_2}) + k,
\end{aligned} \tag{2.13}$$

where l_2 is the smallest positive integer such that kl_2 is a perfect square.

Proof. Let (x, y, z) be a triple where $z - x = k$. By the general formula for primitive Pythagorean triples,

$$\begin{aligned}
x &= l(2mn) \\
y &= l(m^2 - n^2) \\
z &= l(m^2 + n^2),
\end{aligned} \tag{2.14}$$

where l is some positive integer and $m > n > 0$ are two relatively prime positive integers of opposite parity. Since $z - x = k$, we have $l(m^2 + n^2) - l(2mn) = k$, which yields $n = m - \sqrt{\frac{k}{l}}$. Substituting this expression for n into equation 2.14, we get:

$$\begin{aligned} x &= 2m(ml - \sqrt{kl}) \\ y &= 2m\sqrt{kl} - k \\ z &= 2m(ml - \sqrt{kl}) + k. \end{aligned} \tag{2.15}$$

Now we need to show that there exists some integer m_2 such that:

$$\begin{aligned} x &= 2m_2(m_2 l_2 - \sqrt{kl_2}) \\ y &= 2m_2\sqrt{kl_2} - k \\ z &= 2m_2(m_2 l_2 - \sqrt{kl_2}) + k, \end{aligned} \tag{2.16}$$

where l_2 is the smallest positive integer such that kl_2 is a perfect square. Let $q = \sqrt{kl_2}$ and let $p = \sqrt{kl}$ (note that p must be an integer for x, y, and z to be integers). Let $2^{a_0}p_1^{a_1}p_2^{a_2}...p_r^{a_r}$ be the prime factorization of k. Also, let $p_{o1}, p_{o2}, ..., p_{oj}$ be the primes with odd exponents and let $p_{e1}, p_{e2}, ..., p_{es}$ be the primes with even exponents. Now let $a_{o1}, a_{o2}, ..., a_{oj}$ be the odd exponents and let $a_{e1}, a_{e2}, ..., a_{es}$ be the even exponents. We will show that q divides p. Now,

$$q = \sqrt{kl_2} = \sqrt{p_0^{a_0}p_1^{a_1}p_2^{a_2}...p_r^{a_r}l_2} = \sqrt{{p_{e1}}^{a_{e1}}{p_{e2}}^{a_{e2}}...{p_{es}}^{a_{es}}{p_{o1}}^{a_{o1}}{p_{o2}}^{a_{o2}}...{p_{oj}}^{a_{oj}}l_2}. \tag{2.17}$$

Since l_2 is the smallest integer that will make every exponent under the radical even, $l_2 = p_{o1}p_{o2}...p_{oj}$, and

$$\sqrt{kl_2} = \sqrt{{p_{e1}}^{a_{e1}}{p_{e2}}^{a_{e2}}...{p_{es}}^{a_{es}}{p_{o1}}^{a_{o1}+1}{p_{o2}}^{a_{o2}+1}...{p_{oj}}^{a_{oj}+1}}. \tag{2.18}$$

Now,

$$p = \sqrt{kl} = \sqrt{p_0^{a_0}p_1^{a_1}p_2^{a_2}...p_r^{a_r}l} = \sqrt{{p_{e1}}^{a_{e1}}{p_{e2}}^{a_{e2}}...{p_{es}}^{a_{es}}{p_{o1}}^{a_{o1}}{p_{o2}}^{a_{o2}}...{p_{oj}}^{a_{oj}}l}. \tag{2.19}$$

Now in order for p to be an integer, l must contain as factors at least one copy of every prime with an odd exponent in the prime factorization of k; thus $l_2|l$, so $l = l_2r$, for some integer r. The integer r must also contain only even exponents in its prime factorization, or p will again not be an integer. Thus $\sqrt{r} = d$, for some integer d. So we have:

$$\sqrt{kl} = \sqrt{{p_{e1}}^{a_{e1}}{p_{e2}}^{a_{e2}}...{p_{es}}^{a_{es}}{p_{o1}}^{a_{o1}+1}{p_{o2}}^{a_{o2}+1}...{p_{oj}}^{a_{oj}+1}r}, \tag{2.20}$$

and therefore

$$\frac{p}{q} = \frac{\sqrt{kl}}{\sqrt{kl_2}} = \frac{\sqrt{{p_{e1}}^{a_{e1}}{p_{e2}}^{a_{e2}}...{p_{es}}^{a_{es}}{p_{o1}}^{a_{o1}+1}{p_{o2}}^{a_{o2}+1}...{p_{oj}}^{a_{oj}+1}r}}{\sqrt{{p_{e1}}^{a_{e1}}{p_{e2}}^{a_{e2}}...{p_{es}}^{a_{es}}{p_{o1}}^{a_{o1}+1}{p_{o2}}^{a_{o2}+1}...{p_{oj}}^{a_{oj}+1}}} = \sqrt{r} = d. \tag{2.21}$$

Thus $q|p$. Now let $m_2 = md$. Then:

$$\begin{aligned} 2m_2(m_2l_2 - \sqrt{kl_2}) &= 2md(mdl_2 - \sqrt{kl_2}) \\ &= 2m^2rl_2 - 2md\sqrt{kl_2} \\ &= 2m^2l - 2m\sqrt{kl_2r} \\ &= 2m^2l - 2m\sqrt{kl} \\ &= 2m(ml - \sqrt{kl}) \\ &= x \end{aligned} \tag{2.22}$$

$$
\begin{aligned}
2m_2\sqrt{kl_2} - k &= 2md\sqrt{kl_2} - k \\
&= 2m\sqrt{kl_2 r} - k \\
&= 2m\sqrt{kl} - k \\
&= y
\end{aligned} \tag{2.23}
$$

$$
\begin{aligned}
2m_2(m_2 l_2 - \sqrt{kl_2}) + k &= 2md(mdl_2 - \sqrt{kl_2}) + k \\
&= 2m(ml - \sqrt{kl}) + k \\
&= z.
\end{aligned}
$$

□

Thus we have shown that for any triple (x, y, z) where $z - x = k$, there exist integers m_2 and l_2 such that

$$
\begin{aligned}
x &= 2m_2(m_2 l_2 - \sqrt{kl_2}) \\
y &= 2m_2\sqrt{kl_2} - k \\
z &= 2m_2(m_2 l_2 - \sqrt{kl_2}) + k,
\end{aligned} \tag{2.24}
$$

where l_2 is the smallest positive integer such that kl_2 is a perfect square.

Now we are trying to show that when k is even and $z - x = k$, our triple $(x, y, z) \in G_k$. In Lemma 3, we established the general form that any triple (x, y, z) where $z - x = k$ must have. We now state an additional lemma that provides a simplification of this form when k is required to be even.

Lemma 4. *Suppose (x, y, z) is a triple such that $z - x = k$ for some even k. Then there exists some integer m_1 such that*

$$
\begin{aligned}
x &= l_1 m_1^2 - \tfrac{k}{2} \\
y &= m_1\sqrt{2l_1 k} \\
z &= l_1 m_1^2 + \tfrac{k}{2},
\end{aligned} \tag{2.25}
$$

where l_1 is the smallest integer such that $\sqrt{2kl_1}$ is a perfect square.

Proof. Let (x, y, z) be a triple such that $z - x = k$ for some even k. Now by Lemma 3, (x, y, z) has the following form:

$$
\begin{aligned}
x &= 2m_2(m_2 l_2 - \sqrt{kl_2}) \\
y &= 2m_2\sqrt{kl_2} - k \\
z &= 2m_2(m_2 l_2 - \sqrt{kl_2}) + k,
\end{aligned} \tag{2.26}
$$

for some integers m_2 and l_2, where l_2 is the smallest positive integer such that kl_2 is a perfect square. Now k is even, and it has either an even or an odd number of 2's in its prime factorization. We will show that in either case, an integer m_1 can be found that will allow us to write the triple in the form given in Lemma 4.

Case 1: Suppose k has an odd number of 2's in its prime factorization. Then $k = {p_{e1}}^{n_{e1}}{p_{e2}}^{n_{e2}}...{p_{es}}^{n_{es}} 2^{n_{o1}}{p_{o2}}^{n_{o2}}...{p_{oj}}^{n_{oj}}$, where $2, p_{o2}, ..., p_{oj}$ are the prime factors with odd exponents and $p_{e1}, p_{e2}, ..., p_{es}$ are the prime factors with even exponents. Recall that since l_1 is the smallest integer such that $2kl_1$ is a perfect square, $l_1 = p_{o2}...p_{oj}$ (since k contains an odd number of 2's, there is no need for an additional factor of 2 to make $2kl_1$ a perfect square; thus 2 is not a factor of l_1). However, l_2 must have a factor of 2 in order for kl_2 to be a perfect square; namely, $l_2 = 2p_{o2}...p_{oj}$. Therefore, $l_1 = \frac{l_2}{2}$, that is $l_2 = 2l_1$. Thus:

$$
2kl_1 = {p_{e1}}^{n_{e1}}{p_{e2}}^{n_{e2}}...{p_{es}}^{n_{es}} 2^{n_{o1}+1}{p_{o2}}^{n_{o2}+1}...{p_{oj}}^{n_{oj}+1}, \tag{2.27}
$$

which implies that

$$\frac{k}{\sqrt{2kl_1}} = \frac{{p_{e1}}^{n_{e1}}{p_{e2}}^{n_{e2}}...{p_{e_s}}^{n_{es}}2^{n_{o1}}{p_{o2}}^{n_{o2}}...{p_{o_j}}^{n_{oj}}}{{p_{e1}}^{\frac{n_{e1}}{2}}{p_{e2}}^{\frac{n_{e2}}{2}}...{p_{e_s}}^{\frac{n_{e_s}}{2}}2^{\frac{n_{o1}+1}{2}}{p_{o2}}^{\frac{n_{o2}+1}{2}}...{p_{o_j}}^{\frac{n_{oj}+1}{2}}} = {p_{e1}}^{\frac{n_{e1}}{2}}{p_{e2}}^{\frac{n_{e2}}{2}}...{p_{e_s}}^{\frac{n_{e_s}}{2}}2^{\frac{n_{o1}-1}{2}}{p_{o2}}^{\frac{n_{o2}-1}{2}}...{p_{o_j}}^{\frac{n_{oj}-1}{2}}. \quad (2.28)$$

Now since $n_{e1}, n_{e2}, ..., n_{es}$ and $n_{o1}-1, n_{o2}-1, ..., n_{oj}-1$ are all even, nonnegative integers, $\frac{n_{e1}}{2}$, $\frac{n_{e2}}{2}$..., $\frac{n_{e_s}}{2}$ and $\frac{n_{o1}-1}{2}$, $\frac{n_{o2}-1}{2}$, ..., $\frac{n_{oj}-1}{2}$ are all nonnegative integers, and thus the above product is an integer. Namely, $\frac{k}{\sqrt{2kl_1}}$ is an integer.

Now let $m_1 = 2m_2 - \frac{k}{\sqrt{2kl_1}}$. We will show that this is the integer we need to write the triple (x, y, z) in the form given in Lemma 4. We have:

$$\begin{aligned} x &= 2m_2(m_2l_2 - \sqrt{kl_2}) \\ &= 2(\tfrac{m_1}{2} + \tfrac{k}{2\sqrt{2kl_1}})(\tfrac{2l_1m_1}{2} + \tfrac{2l_1k}{2\sqrt{2kl_1}} - \sqrt{2kl_1}) \\ &= (m_1 + \tfrac{k}{\sqrt{2kl_1}})(l_1m_1 + \tfrac{l_1k}{\sqrt{2kl_1}} - \sqrt{2kl_1}) \\ &= (m_1 + \tfrac{\sqrt{k}}{\sqrt{2l_1}})(l_1m_1 + \tfrac{\sqrt{l_1k}}{\sqrt{2}} - \sqrt{2kl_1}) \\ &= l_1{m_1}^2 + m_1\tfrac{\sqrt{l_1k}}{\sqrt{2}} - m_1\sqrt{2kl_1} + m_1\tfrac{\sqrt{l_1k}}{\sqrt{2}} + \tfrac{k\sqrt{l_1}}{2\sqrt{l_1}} - k \\ &= l_1{m_1}^2 + m_1(\tfrac{\sqrt{l_1k}}{\sqrt{2}} - \tfrac{\sqrt{2}\sqrt{2kl_1}}{\sqrt{2}} + \tfrac{\sqrt{l_1k}}{\sqrt{2}}) - \tfrac{k}{2} \\ &= l_1{m_1}^2 + m_1(\tfrac{2\sqrt{l_1k}-2\sqrt{l_1k}}{\sqrt{2}}) - \tfrac{k}{2} \\ &= l_1{m_1}^2 - \tfrac{k}{2} \end{aligned} \quad (2.29)$$

$$\begin{aligned} y &= 2m_2\sqrt{kl_2} - k \\ &= 2(\tfrac{m_1}{2} + \tfrac{k}{2\sqrt{2kl_1}})(\sqrt{2kl_1}) - k \\ &= m_1\sqrt{2kl_1} \end{aligned}$$

$$\begin{aligned} z &= 2m_2(m_2l_2 - \sqrt{kl_2}) + k \\ &= l_1{m_1}^2 - \tfrac{k}{2} + k \\ &= l_1{m_1}^2 + \tfrac{k}{2}, \end{aligned}$$

since we just showed that $x = 2m_2(m_2l_2 - \sqrt{kl_2}) = l_1{m_1}^2 - \frac{k}{2}$. Thus when there are an odd number of 2's in the prime factorization of k, a triple (x, y, z) where $z - x = k$ can be written in the form given in Lemma 4. Now we proceed to the second case in the proof of Lemma 4.

Case 2. Suppose k has an even number of 2's in its prime factorization. We will show that $m_1 = m_2 - \frac{k}{\sqrt{2kl_1}}$ is the integer that will allow us to write (x, y, z) in the desired form. First of all, however, we must show that $\frac{k}{\sqrt{2kl_1}}$ is indeed an integer.

Since k has an even number of 2's in its prime factorization, $k = 2^{n_{e1}}{p_{e2}}^{n_{e2}}...{p_{e_s}}^{n_{es}}{p_{o1}}^{n_{o1}}{p_{o2}}^{n_{o2}}...{p_{o_j}}^{n_{oj}}$, where $p_{o1}, p_{o2}, ..., p_{oj}$ are the prime factors with odd exponents and $2, p_{e2}, ..., p_{es}$ are the prime factors with even exponents. Now since k contains an even number of 2's, l_1 must contain a factor of 2 in order for $2kl_1$ to be a perfect square. However, l_2 does not need a factor of 2 in order for l_2k to be a perfect square, and thus $l_1 = 2l_2$. So we get $l_1 = 2p_{o1}p_{o2}...p_{oj}$ and $2kl_1 = 2^{n_{e1}+2}{p_{e2}}^{n_{e2}}...{p_{es}}^{n_{es}}{p_{o1}}^{n_{o1}+1}{p_{o2}}^{n_{o2}+1}...{p_{oj}}^{n_{oj}+1}$. Thus:

$$\frac{k}{\sqrt{2kl_1}} = \frac{2^{n_{e1}}{p_{e2}}^{n_{e2}}...{p_{es}}^{n_{es}}{p_{o1}}^{n_{o1}}{p_{o2}}^{n_{o2}}...{p_{oj}}^{n_{oj}}}{\sqrt{2^{n_{e1}+2}{p_{e2}}^{n_{e2}}...{p_{es}}^{n_{es}}{p_{o1}}^{n_{o1}+1}{p_{o2}}^{n_{o2}+1}...{p_{oj}}^{n_{oj}+1}}}$$

$$= \frac{2^{n_{e1}}{p_{e2}}^{n_{e2}}...{p_{es}}^{n_{es}}{p_{o1}}^{n_{o1}}{p_{o2}}^{n_{o2}}...{p_{oj}}^{n_{oj}}}{2^{\frac{n_{e1}+2}{2}}{p_{e2}}^{\frac{n_{e2}}{2}}...{p_{es}}^{\frac{n_{es}}{2}}{p_{o1}}^{\frac{n_{o1}+1}{2}}{p_{o2}}^{\frac{n_{o2}+1}{2}}...{p_{oj}}^{\frac{n_{oj}+1}{2}}}$$

$$= 2^{\frac{n_{e1}-2}{2}}{p_{e2}}^{\frac{n_{e2}}{2}}...{p_{es}}^{\frac{n_{es}}{2}}{p_{o1}}^{\frac{n_{o1}-1}{2}}{p_{o2}}^{\frac{n_{o2}-1}{2}}...{p_{oj}}^{\frac{n_{oj}-1}{2}}.$$

Now $n_{e1}-2, n_{e2}, ..., n_{es}$ and $n_{o1}-1, n_{o2}-1, ..., n_{oj}-1$ are all even; thus all the exponents are integers. Note that since k is even, $n_{e1} \geq 2$, and thus $n_{e1}-2 \geq 0$. Thus we know that all of the exponents are in fact nonnegative integers, and thus the above product is an integer; namely, $\frac{k}{\sqrt{2kl_1}}$ is an integer.
Now let $m_1 = m_2 - \frac{k}{\sqrt{2kl_1}}$. Then

$$\begin{aligned}
x &= 2m_2(m_2l_2 - \sqrt{kl_2}) \\
&= 2(m_1 + \tfrac{k}{\sqrt{2kl_1}})(\tfrac{l_1m_1}{2} + \tfrac{l_1k}{2\sqrt{2kl_1}} - \sqrt{\tfrac{kl_1}{2}}) \\
&= (2m_1 + \tfrac{\sqrt{2k}}{\sqrt{l_1}})(\tfrac{l_1m_1}{2} + \tfrac{\sqrt{l_1k}}{2\sqrt{2}} - \tfrac{\sqrt{l_1k}}{\sqrt{2}}) \\
&= l_1m_1^2 + m_1\tfrac{\sqrt{l_1k}}{\sqrt{2}} - m_1\sqrt{2kl_1} + m_1\tfrac{\sqrt{2kl_1}}{2} + \tfrac{k}{2} - k \\
&= l_1m_1^2 + m_1(\tfrac{\sqrt{2l_1k}}{2} - \tfrac{2\sqrt{2kl_1}}{2} + \tfrac{\sqrt{2kl_1}}{2}) - \tfrac{k}{2} \\
&= l_1m_1^2 - \tfrac{k}{2} \\
\\
y &= 2m_2\sqrt{kl_2} - k \\
&= 2(m_1 + \tfrac{k}{\sqrt{2kl_1}})(\sqrt{\tfrac{kl_1}{2}}) - k \\
&= (2m_1 + \tfrac{2k}{\sqrt{2kl_1}})(\sqrt{\tfrac{kl_1}{2}}) - k \\
&= m_1\sqrt{2kl_1} + k - k \\
&= m_1\sqrt{2kl_1} \\
\\
z &= 2m_2(m_2l_2 - \sqrt{kl_2}) + k \\
&= l_1m_1^2 - \tfrac{k}{2} + k \\
&= l_1m_1^2 + \tfrac{k}{2},
\end{aligned} \tag{2.30}$$

since we already showed in our analysis of x that $2m_2(m_2l_2 - \sqrt{kl_2}) = l_1m_1^2 - \frac{k}{2}$. □

Thus we have shown that when (x, y, z) is a triple where $z - x = k$ for some even k, there exists some integer m_1 such that the triple can be written as:

$$x = l_1 m_1^2 - \tfrac{k}{2}$$
$$y = m_1\sqrt{2l_1k} \tag{2.31}$$
$$z = l_1 m_1^2 + \tfrac{k}{2},$$

where l_1 is the smallest integer such that $2kl_1$ is a perfect square. We are trying to show that when (x, y, z) is a triple where $z - x = k$ for some even k, $(x, y, z) \in G_k$. Having shown in Lemma 4 that (x, y, z) has the above form, we can now use a derivation almost identical to the one in 2.12 to show that $(x, y, z) \in G_k$; namely, we will show that when the integer m_1 is inserted into the formula given in Theorem 1, the triple (x, y, z) is produced. Note that since $x > 0$ and $x = l_1m_1^2 - \frac{k}{2}$, we have $l_1m_1^2 - \frac{k}{2} > 0$, and an identical argument to the one given in 2.11 shows that $m_1 > \lfloor \frac{f(k)}{2} \rfloor$. Thus m_1 is in the domain of the formula given in Theorem 1, and we have:

$$\begin{aligned}
x &= l_1m_1^2 - \tfrac{k}{2} \\
&= \tfrac{2k}{f(k)^2} m_1^2 - \tfrac{2k}{4} \\
&= \tfrac{2k}{f(k)^2} (m_1^2 - \tfrac{f(k)^2}{4}) \\
&= \tfrac{2k}{f(k)^2} (m_1 - \tfrac{f(k)}{2})(m_1 + \tfrac{f(k)}{2}) \\
&= \tfrac{2k}{f(k)^2} (m_1 - \lfloor \tfrac{f(k)}{2} \rfloor)(m_1 + \lceil \tfrac{f(k)}{2} \rceil)
\end{aligned}$$

$$\begin{aligned}
y &= m_1\sqrt{2l_1k} \\
&= m_1\sqrt{\tfrac{2k2k}{f(k)^2}} \\
&= m_1\tfrac{2k}{f(k)} \\
&= \tfrac{k}{f(k)}2m_1 \\
&= \tfrac{k}{f(k)}(2m_1 + k \bmod 2)
\end{aligned} \tag{2.32}$$

$$\begin{aligned}
z &= l_1m_1^2 + \tfrac{k}{2} \\
&= l_1m_1^2 - \tfrac{k}{2} + k \\
&= \tfrac{2k}{f(k)^2} (m_1 - \lfloor \tfrac{f(k)}{2} \rfloor)(m_1 + \lceil \tfrac{f(k)}{2} \rceil) + k.
\end{aligned}$$

Thus we see that when the integer m_1 is inserted into the formula given in Theorem 1, the triple (y, x, z) is produced. Since we are considering the triple (y, x, z) to be the same triple as (x, y, z), $(x, y, z) \in G_k$. In the above derivation we once again made use of the fact that $l_1 = \frac{2k}{f(k)^2}$, which we proved in Lemma 2.
Thus we have shown that when k is even, $(x, y, z) \in G_k$ for the subcases $z - y = k$ and $z - x = k$. Since these are the only possible subcases, we have shown that when k is even, $(x, y, z) \in G_k$. We now proceed to Case 2, where k is odd.

Case 2: Suppose that k is an odd integer. Now by the general formula for primitive Pythagorean triples, we know that (x, y, z) has the following form:

$$x = l(2mn)$$
$$y = l(m^2 - n^2) \tag{2.33}$$
$$z = l(m^2 + n^2),$$

where l is some positive integer and $m > n > 0$ are two relatively prime positive integers of opposite parity. Now since we are supposing this triple has difference k, either $z - x = k$ or $z - y = k$. Observe that $z - y = l(m^2 + n^2) - l(m^2 - n^2) = 2ln^2$; thus $z - y$ is even. Since k is odd, $z - y \neq k$, and we must have $z - x = k$. Thus for k odd, we do not have two subcases like we did for k even. Now by Lemma 3 (since Lemma 3 holds for both even and odd k), the triple (x, y, z) can be written in the following form for some integer m_2:

$$x = 2m_2(m_2 l_2 - \sqrt{kl_2})$$
$$y = 2m_2\sqrt{kl_2} - k \tag{2.34}$$
$$z = 2m_2(m_2 l_2 - \sqrt{kl_2}) + k,$$

where l_2 is the smallest positive integer such that kl_2 is a perfect square. We need to show that this triple is in the set G_k. We will proceed by showing that the triple $(y, x, z) \in G_k$, which clearly implies that $(x, y, z) \in G_k$. To show $(y, x, z) \in G_k$, we must show that there exists some integer m_1 greater than $\lfloor \frac{f(k)}{2} \rfloor$ such that:

$$y = \tfrac{k}{f(k)}(2m_1 + k \bmod 2)$$
$$x = \tfrac{2k}{f(k)^2}(m_1 - \lfloor \tfrac{f(k)}{2} \rfloor)(m_1 + \lceil \tfrac{f(k)}{2} \rceil) \tag{2.35}$$
$$z = \tfrac{2k}{f(k)^2}(m_1 - \lfloor \tfrac{f(k)}{2} \rfloor)(m_1 + \lceil \tfrac{f(k)}{2} \rceil) + k.$$

Recall that l_1 is the smallest positive integer such that $2kl_1$ is a perfect square and that l_2 is the smallest integer such that kl_2 is a perfect square. Thus l_2 contains one copy of every prime factor of k that has an odd exponent, while l_1 contains one copy of every prime factor of $2k$ that has an odd exponent. Since 2 is not a factor of k, l_2 does not contain a factor of 2, but since 2 is a factor of $2k$ and has an odd exponent in the prime factorization of $2k$, l_1 does have a factor of 2. However, all the other factors of l_1 and l_2 are clearly the same. Thus we have $l_2 = \frac{l_1}{2}$. Recall that $l_1 = \frac{2k}{f(k)^2}$, as shown in Lemma 2, which holds regardless of the parity of k. So:

$$l_2 = \frac{l_1}{2} = \frac{2k}{2f(k)^2} = \frac{k}{f(k)^2}. \tag{2.36}$$

Now let $m_1 = m_2 - \frac{f(k)+1}{2}$. We will show that m_1 is the integer that will produce the triple (y, x, z) when inserted into the formula given in Theorem 1. First of all, however, we must show that m_1 is in the domain of the formula given in Theorem 1; namely, we must verify that m_1 is an integer and that $m_1 > \lfloor \frac{f(k)}{2} \rfloor$. Now since k is odd, there is only one 2 in the prime fctorization of $2k$, that is $2k = 2^1 p_1^{n_1} p_2^{n_2} ... p_r^{n_r}$, where $2, p_1, p_2, ..., p_r$ are the distinct prime factors of $2k$. Thus $f(k) = 2^{\lfloor \frac{1}{2} \rfloor} p_1^{\lfloor \frac{n_1}{2} \rfloor} p_2^{\lfloor \frac{n_2}{2} \rfloor} ... p_r^{\lfloor \frac{n_r}{2} \rfloor}$, which shows that $f(k)$ has no factors of 2. Thus $f(k)$ is odd, making $f(k) + 1$ even and $\frac{f(k)+1}{2}$ an integer. Therefore, $m_1 = m_2 - \frac{f(k)+1}{2}$ is also an integer.

Note also that since $x > 0$ and $x = 2m_2(m_2 l_2 - \sqrt{kl_2})$, we must have $m_2 l_2 - \sqrt{kl_2} > 0$. Thus:

$$
\begin{aligned}
&(m_1 + \tfrac{f(k)+1}{2})l_2 > \sqrt{kl_2} \\
&(m_1 + \tfrac{f(k)+1}{2})\tfrac{k}{f(k)^2} > \sqrt{\tfrac{k^2}{f(k)^2}} \\
&m_1 + \tfrac{f(k)+1}{2} > f(k) \\
&m_1 > \tfrac{2f(k)}{2} - \tfrac{f(k)+1}{2} \\
&m_1 > \tfrac{f(k)-1}{2} \\
&m_1 > \lfloor \tfrac{f(k)}{2} \rfloor \qquad \text{(since } k \text{ is odd, } f(k) \text{ is odd, and thus } \tfrac{f(k)-1}{2} = \lfloor \tfrac{f(k)}{2} \rfloor\text{).}
\end{aligned}
\tag{2.37}
$$

Thus m_1 is an integer greater than $\lfloor \frac{f(k)}{2} \rfloor$ and is thus in the domain of the formula given in Theorem 1. Since $m_2 = m_1 + \frac{f(k)+1}{2}$, we get:

$$
\begin{aligned}
y &= 2m_2\sqrt{kl_2} - k \\
&= 2m_2\sqrt{\tfrac{k^2}{f(k)^2}} - k \\
&= 2m_2\tfrac{k}{f(k)} - k \\
&= 2(m_1 + \tfrac{f(k)+1}{2})(\tfrac{k}{f(k)}) - k \\
&= (2m_1 + f(k) + 1)(\tfrac{k}{f(k)}) - k \\
&= \tfrac{2k}{f(k)}m_1 + k + \tfrac{k}{f(k)} - k \\
&= \tfrac{k}{f(k)}(2m_1 + 1) \\
&= \tfrac{k}{f(k)}(2m_1 + k \bmod 2) \\
x &= 2m_2(m_2 l_2 - \sqrt{kl_2}) \\
&= 2(m_1 + \tfrac{f(k)+1}{2})[(m_1 + \tfrac{f(k)+1}{2})\tfrac{k}{f(k)^2} - \sqrt{\tfrac{k^2}{f(k)^2}}] \\
&= (2m_1 + f(k) + 1)(\tfrac{m_1 k}{f(k)^2} + \tfrac{k}{2f(k)} + \tfrac{k}{2f(k)^2} - \tfrac{k}{f(k)}) \\
&= (2m_1 + f(k) + 1)(\tfrac{m_1 k}{f(k)^2} + \tfrac{k}{2f(k)^2} - \tfrac{k}{2f(k)}) \\
&= m_1^2\tfrac{2k}{f(k)^2} + m_1\tfrac{k}{f(k)^2} - m_1\tfrac{k}{f(k)} + m_1\tfrac{k}{f(k)} + \tfrac{k}{2f(k)} - \tfrac{k}{2} + m_1\tfrac{k}{f(k)^2} + \tfrac{k}{2f(k)^2} - \tfrac{k}{2f(k)} \\
&= m_1^2\tfrac{2k}{f(k)^2} + m_1\tfrac{2k}{f(k)^2} - \tfrac{k}{2} + \tfrac{k}{2f(k)^2} \\
&= \tfrac{2k}{f(k)^2}(m_1^2 + m_1 - \tfrac{f(k)^2}{4} + \tfrac{1}{4}) \\
&= \tfrac{2k}{f(k)^2}(m_1^2 + m_1(\tfrac{f(k)+1}{2} - \tfrac{f(k)-1}{2}) - (\tfrac{f(k)^2-1}{4})) \\
&= \tfrac{2k}{f(k)^2}(m_1^2 - m_1(\tfrac{f(k)-1}{2}) + m_1(\tfrac{f(k)+1}{2}) - (\tfrac{f(k)-1}{2})(\tfrac{f(k)+1}{2}))
\end{aligned}
\tag{2.38}
$$

Now since k is odd, $f(k)$ is odd, and thus we have that $\lfloor \frac{f(k)}{2} \rfloor = \frac{f(k)-1}{2}$ and $\lceil \frac{f(k)}{2} \rceil = \frac{f(k)+1}{2}$. Substituting this we get:

$$\begin{aligned} x &= \tfrac{2k}{f(k)^2}(m_1^2 - m_1\lfloor \tfrac{f(k)}{2} \rfloor + m_1\lceil \tfrac{f(k)}{2} \rceil - \lceil \tfrac{f(k)}{2} \rceil \lfloor \tfrac{f(k)}{2} \rfloor) \\ &= \tfrac{2k}{f(k)^2}(m_1 - \lfloor \tfrac{f(k)}{2} \rfloor)(m_1 + \lceil \tfrac{f(k)}{2} \rceil). \end{aligned} \tag{2.39}$$

Likewise, we see that:

$$z = x + k = \frac{2k}{f(k)^2}(m_1 - \lfloor \frac{f(k)}{2} \rfloor)(m_1 + \lceil \frac{f(k)}{2} \rceil) + k. \tag{2.40}$$

Thus when (x, y, z) is a triple where $z - x = k$ for some odd integer k, (y, x, z) is clearly an element of G_k because it is the triple that is generated when the integer m_1 is inserted into the formula given in Theorem 1. Since, again, we are considering (y, x, z) to be the same triple as (x, y, z), we have shown that $(x, y, z) \in G_k$.

Thus we have shown that for an arbitrary triple (x, y, z) of difference k, that is for an arbitrary element of S_k, $(x, y, z) \in G_k$ both when k is even and when k is odd. Therefore we have shown that $S_k \subseteq G_k$. For k even, we examined two subcases: $z - y = k$ and $z - x = k$, while for k odd we had only $z - x = k$.
To conclude the proof of our main theorem, we must now show that $G_k \subseteq S_k$. Let (q, r, s) be an arbitrary element of G_k and let m and k be the integers required by Theorem 1 to produce (q, r, s). In order for $(q, r, s) \in S_k$, we need to verify that (q, r, s) is a Pythagorean triple of difference k. Note that if (q, r, s) is a Pythagorean triple, then it is clearly a Pythagorean triple with a difference of k. For since (q, r, s) has the following form:

$$\begin{aligned} q &= \tfrac{k}{f(k)}(2m + k \bmod 2) \\ r &= \tfrac{2k}{f(k)^2}(m - \lfloor \tfrac{f(k)}{2} \rfloor)(m + \lceil \tfrac{f(k)}{2} \rceil) \\ s &= \tfrac{2k}{f(k)^2}(m - \lfloor \tfrac{f(k)}{2} \rfloor)(m + \lceil \tfrac{f(k)}{2} \rceil) + k, \end{aligned} \tag{2.41}$$

clearly $s - r = k$. Therefore, all that needs to be shown is that (q, r, s) is a Pythagorean triple, namely that q, r, and s are positive integers and that $q^2 + r^2 = s^2$.
Now since $m > \lfloor \frac{f(k)}{2} \rfloor$, clearly q, r, and s must all be positive. Thus if $\frac{k}{f(k)}$ and $\frac{2k}{f(k)^2}$ are integers, then q, r, and s must in fact be positive integers. It is clear from Lemma 2 that $\frac{2k}{f(k)^2}$ is an integer; thus we will simply show that $\frac{k}{f(k)}$ is an integer. We will look at the k even and k odd cases.

Case 1: Suppose k is even and let $2^{n_0}p_1^{n_1}...p_r^{n_r}$ be the prime factorization of k. Then:

$$\begin{aligned} \tfrac{k}{f(k)} &= \frac{2^{n_0}p_1^{n_1}...p_r^{n_r}}{2^{\lfloor \frac{n_0+1}{2} \rfloor}p_1^{\lfloor \frac{n_1}{2} \rfloor}p_2^{\lfloor \frac{n_2}{2} \rfloor}...p_r^{\lfloor \frac{n_r}{2} \rfloor}} \\ &= 2^{n_0 - \lfloor \frac{n_0+1}{2} \rfloor}p_1^{n_1 - \lfloor \frac{n_1}{2} \rfloor}p_2^{n_2 - \lfloor \frac{n_2}{2} \rfloor}...p_r^{n_r - \lfloor \frac{n_r}{2} \rfloor}. \end{aligned} \tag{2.42}$$

Now since every exponent in the above expression is clearly nonnegative, $\frac{k}{f(k)}$ is an integer.

Case 2: Suppose k is odd and that $k = p_1^{n_1}p_2^{n_2}...p_r^{n_r}$ is the prime factorization of k. Then:

$$\begin{aligned} \tfrac{k}{f(k)} &= \frac{p_1^{n_1}p_2^{n_2}...p_r^{n_r}}{2^{\lfloor \frac{1}{2} \rfloor}p_1^{\lfloor \frac{n_1}{2} \rfloor}p_2^{\lfloor \frac{n_2}{2} \rfloor}...p_r^{\lfloor \frac{n_r}{2} \rfloor}} \\ &= \frac{p_1^{n_1}p_2^{n_2}...p_r^{n_r}}{2^0 p_1^{\lfloor \frac{n_1}{2} \rfloor}p_2^{\lfloor \frac{n_2}{2} \rfloor}...p_r^{\lfloor \frac{n_r}{2} \rfloor}} \\ &= p_1^{n_1 - \lfloor \frac{n_1}{2} \rfloor}p_2^{n_2 - \lfloor \frac{n_2}{2} \rfloor}...p_r^{n_r - \lfloor \frac{n_r}{2} \rfloor} \end{aligned} \tag{2.43}$$

Once again, since every exponent is clearly a nonnegative integer, $\frac{k}{f(k)}$ is also an integer.
Thus we have verified that q, r, and s are indeed positive integers. Now we will show that $q^2 + r^2 = s^2$. Once again, we will look at two cases.

Case 1: Suppose that k is even. Then since k is even, $f(k)$ is also even because it clearly contains a factor of 2. Thus $\lfloor \frac{f(k)}{2} \rfloor = \lceil \frac{f(k)}{2} \rceil = \frac{f(k)}{2}$. Also, $k \bmod 2 = 0$. Thus our formulas for q, r, and s can be simplified as follows:

$$\begin{aligned} q &= \tfrac{k}{f(k)}(2m) \\ r &= \tfrac{2k}{f(k)^2}(m - \tfrac{f(k)}{2})(m + \tfrac{f(k)}{2}) \\ s &= \tfrac{2k}{f(k)^2}(m - \tfrac{f(k)}{2})(m + \tfrac{f(k)}{2}) + k. \end{aligned} \tag{2.44}$$

Thus:

$$\begin{aligned} q^2 + r^2 &= (2m\tfrac{k}{f(k)})^2 + [\tfrac{2k}{f(k)^2}(m - \tfrac{f(k)}{2})(m + \tfrac{f(k)}{2})]^2 \\ &= (2m\tfrac{k}{f(k)})^2 + [\tfrac{2k}{f(k)^2}(m^2 - \tfrac{f(k)^2}{4})]^2 \\ &= (2m\tfrac{k}{f(k)})^2 + (\tfrac{2k}{f(k)^2}m^2 - \tfrac{k}{2})^2 \\ &= 4m^2\tfrac{k^2}{f(k)^2} + \tfrac{4k^2}{f(k)^4}m^4 - \tfrac{2k^2}{f(k)^2}m^2 + \tfrac{k^2}{4} \\ &= \tfrac{4k^2}{f(k)^4}m^4 + \tfrac{2k^2}{f(k)^2}m^2 + \tfrac{k^2}{4} \\ &= (\tfrac{2k}{f(k)^2}m^2 + \tfrac{k}{2})^2 \\ &= (\tfrac{2k}{f(k)^2}m^2 - \tfrac{k}{2} + k)^2 \\ &= [\tfrac{2k}{f(k)^2}(m^2 - \tfrac{f(k)^2}{4}) + k]^2 \\ &= [\tfrac{2k}{f(k)^2}(m - \tfrac{f(k)}{2})(m + \tfrac{f(k)}{2}) + k]^2 \\ &= s^2 \end{aligned} \tag{2.45}$$

Case 2: Suppose that k is odd. Then $f(k)$ is also odd since it does not contain a factor of 2. Thus $\lfloor \frac{f(k)}{2} \rfloor = \frac{f(k)-1}{2}$ and $\lceil \frac{f(k)}{2} \rceil = \frac{f(k)+1}{2}$. Also, $k \bmod 2 = 1$. Thus our formulas for q, r, and s simplify as follows:

$$\begin{aligned} q &= \tfrac{k}{f(k)}(2m + 1) \\ r &= \tfrac{2k}{f(k)^2}(m - \tfrac{f(k)-1}{2})(m + \tfrac{f(k)+1}{2}) \\ s &= \tfrac{2k}{f(k)^2}(m - \tfrac{f(k)-1}{2})(m + \tfrac{f(k)+1}{2}) + k. \end{aligned} \tag{2.46}$$

Thus:

$$\begin{aligned} q^2 + r^2 &= [\tfrac{k}{f(k)}(2m+1)]^2 + [\tfrac{2k}{f(k)^2}(m - \tfrac{f(k)-1}{2})(m + \tfrac{f(k)+1}{2})]^2 \\ &= \tfrac{k^2}{f(k)^2}(4m^2 + 4m + 1) + \tfrac{4k^2}{f(k)^4}[(m - \tfrac{f(k)-1}{2})(m + \tfrac{f(k)+1}{2})]^2 \\ &= \tfrac{4k^2}{f(k)^2}m^2 + \tfrac{4k^2}{f(k)^2}m + \tfrac{k^2}{f(k)^2} + \tfrac{4k^2}{f(k)^4}[(m - \tfrac{f(k)-1}{2})(m + \tfrac{f(k)+1}{2})]^2 \\ &= \tfrac{4k^2}{f(k)^2}m^2 + \tfrac{4k^2}{f(k)^2}m - k^2 + \tfrac{k^2}{f(k)^2} + k^2 + \tfrac{4k^2}{f(k)^4}[(m - \tfrac{f(k)-1}{2})(m + \tfrac{f(k)+1}{2})]^2 \end{aligned} \tag{2.47}$$

$$= \tfrac{4k^2}{f(k)^2}(m^2 + m - \tfrac{f(k)^2}{4} + \tfrac{1}{4}) + k^2 + \tfrac{4k^2}{f(k)^4}[(m - \tfrac{f(k)-1}{2})(m + \tfrac{f(k)+1}{2})]^2$$

$$= \tfrac{4k^2}{f(k)^2}(m - \tfrac{f(k)-1}{2})(m + \tfrac{f(k)+1}{2}) + k^2 + \tfrac{4k^2}{f(k)^4}[(m - \tfrac{f(k)-1}{2})(m + \tfrac{f(k)+1}{2})]^2$$

$$= \tfrac{4k^2}{f(k)^4}[(m - \tfrac{f(k)-1}{2})(m + \tfrac{f(k)+1}{2})]^2 + \tfrac{4k^2}{f(k)^2}(m - \tfrac{f(k)-1}{2})(m + \tfrac{f(k)+1}{2}) + k^2 \tag{2.48}$$

$$= [\tfrac{2k}{f(k)^2}(m - \tfrac{f(k)-1}{2})(m + \tfrac{f(k)+1}{2}) + k]^2$$

$$= s^2$$

So we see that (q, r, s), both for k even and k odd, is indeed a Pythagorean triple with a difference of k, and so $(q, r, s) \in S_k$. Thus we have shown that $G_k \subseteq S_k$. Finally, since $S_k \subseteq G_k$ and $G_k \subseteq S_k$, $G_k = S_k$, which concludes the proof of our main theorem. Thus the formula given in Theorem 1 produces every triple of difference k for any positive integer k.

3 Alternate Formula for Pythagorean Triples of Difference k

Having proved that $G_k = S_k$, we can now prove some additional properties about S_k using this equivalence. In particular, we have the following result:

Theorem 2. *The set of all triples of difference k can be generated by the union of $f(k)$ disjoint, infinite subsets, where the following property holds for each of these subsets:*

$$\frac{d}{dn}Leg2(n) = 2(Leg1(n)), \tag{3.1}$$

where $(Leg1(n),\ Leg2(n),\ Leg2(n) + k)$ denote the generating formulas of the subset. In particular, for k even:

$$\begin{aligned} S_k = \textstyle\bigcup_{i=0}^{f(k)-1}\{(2kn + \tfrac{2ki}{f(k)},\ 2kn^2 + \tfrac{4ki}{f(k)}n + \tfrac{2ki^2}{f(k)^2} - \tfrac{k}{2},\\ 2kn^2 + \tfrac{4ki}{f(k)}n + \tfrac{2ki^2}{f(k)^2} + \tfrac{k}{2})\ : n \in Z, n > \tfrac{1}{2} - \tfrac{i}{f(k)}\}. \end{aligned} \tag{3.2}$$

Similarly, for k odd:

$$\begin{aligned} S_k = \textstyle\bigcup_{i=0}^{f(k)-1}\{(2kn + \tfrac{2ik}{f(k)} + \tfrac{k}{f(k)},\ 2kn^2 + \tfrac{4ki}{f(k)}n + \tfrac{2k}{f(k)}n + \tfrac{2ki^2}{f(k)^2} + \tfrac{2ki}{f(k)^2} - \tfrac{k}{2} + \tfrac{1}{4},\\ 2kn^2 + \tfrac{4ki}{f(k)}n + \tfrac{2k}{f(k)}n + \tfrac{2ki^2}{f(k)^2} + \tfrac{2ki}{f(k)^2} + \tfrac{k}{2} + \tfrac{1}{4})\ : n \in Z, n > \tfrac{1}{2} - \tfrac{1}{2f(k)} - \tfrac{i}{f(k)}\}. \end{aligned} \tag{3.3}$$

The proof of this theorem is a clear application of Theorem 1. Once again, we will examine both the k even and k odd cases.

Proof. Let k be some positive integer. Now by Theorem 1, every triple of difference k is generated by the following formula as m ranges over the positive integers greater than $\lfloor \frac{f(k)}{2} \rfloor$:

$$(\frac{k}{f(k)}(2m + k \bmod 2), \quad \frac{2k}{f(k)^2}(m - \lfloor\frac{f(k)}{2}\rfloor)(m + \lceil\frac{f(k)}{2}\rceil), \quad \frac{2k}{f(k)^2}(m - \lfloor\frac{f(k)}{2}\rfloor)(m + \lceil\frac{f(k)}{2}\rceil) + k). \tag{3.4}$$

Case 1: Suppose k is even. Then our generating formula simplifies to:

$$(\frac{k}{f(k)}(2m), \quad \frac{2k}{f(k)^2}(m - \frac{f(k)}{2})(m + \frac{f(k)}{2}), \quad \frac{2k}{f(k)^2}(m - \frac{f(k)}{2})(m + \frac{f(k)}{2}) + k). \tag{3.5}$$

Thus as m ranges over the positive integers greater than $\lfloor \frac{f(k)}{2} \rfloor$ every triple of difference k is produced. It is easy to see that every distinct positive integer produces a unique triple when inserted into the formula. Now by the division algorithm [2], we can partition all the positive integers greater than $\lfloor \frac{f(k)}{2} \rfloor$ into their residue classes mod $f(k)$. If we were partitioning all of the positive integers into their residue classes mod $f(k)$, the ith residue class would be given by $f(k)n + i$, where n is any nonnegative integer. However, to eliminate producing integers in the ith residue class that are smaller than $\lfloor \frac{f(k)}{2} \rfloor$, we must have $f(k)n + i > \lfloor \frac{f(k)}{2} \rfloor$, which happens exactly when $n > \frac{1}{2} - \frac{i}{f(k)}$. Thus the set $\{f(k)n + i : n \in Z, n > \frac{1}{2} - \frac{i}{f(k)}\}$ is exactly the ith residue class mod $f(k)$ of the positive integers greater than $\lfloor \frac{f(k)}{2} \rfloor$. The restriction on n causes n to start at 0 or 1 as necessary, so that an integer smaller than $\lfloor \frac{f(k)}{2} \rfloor$ is never produced. Now running m over this particular residue class will produce a subset of S_k. In order to only produce the triples generated by the ith residue class, we must substitute $f(k)n+i$ for m in the formula given in Theorem 1 and allow n to range over the integers greater than $\frac{1}{2} - \frac{i}{f(k)}$. After making this substitution into our formula, we find that

$$(2kn + \frac{2ki}{f(k)},\ 2kn^2 + \frac{4ki}{f(k)}n + \frac{2ki^2}{f(k)^2} - \frac{k}{2},\ 2kn^2 + \frac{4ki}{f(k)}n + \frac{2ki^2}{f(k)^2} + \frac{k}{2}) \tag{3.6}$$

gives every triple generated by the ith residue class of the domain of Theorem 1 as n ranges over the integers greater than $\frac{1}{2} - \frac{i}{f(k)}$. Now since every triple produced by our original formula is unique and the residue classes are disjoint, the subsets produced by these residue classes will clearly be disjoint. Also, if we take the union of the triples generated by every residue class we will have the triples generated by every positive integer greater than $\lfloor \frac{f(k)}{2} \rfloor$, that is, we will have the set of all triples of difference k. Thus we find that, for k even:

$$\begin{aligned} S_k = \textstyle\bigcup_{i=0}^{f(k)-1} \{ &(2kn + \tfrac{2ki}{f(k)},\ 2kn^2 + \tfrac{4ki}{f(k)}n + \tfrac{2ki^2}{f(k)^2} - \tfrac{k}{2}, \\ &2kn^2 + \tfrac{4ki}{f(k)}n + \tfrac{2ki^2}{f(k)^2} + \tfrac{k}{2})\ : n \in Z, n > \tfrac{1}{2} - \tfrac{i}{f(k)}\}, \end{aligned} \tag{3.7}$$

as stated in Theorem 2. Thus we have rewritten S_k as the union of $f(k)$ disjoint, infinite subsets. Notice also that, allowing $Leg1(n) = 2kn + \frac{2ki}{f(k)}$ and $Leg2(n) = 2kn^2 + \frac{4ki}{f(k)}n + \frac{2ki^2}{f(k)^2} - \frac{k}{2}$ to denote the generating formulas for the two legs in the ith residue class, the following property holds for each of the subsets in the above union:

$$\begin{aligned} \tfrac{d}{dn}Leg2(n) &= \tfrac{d}{dn}(2kn^2 + \tfrac{4ki}{f(k)}n + \tfrac{2ki^2}{f(k)^2} - \tfrac{k}{2}) \\ &= 4kn + \tfrac{4ki}{f(k)} \\ &= 2(2kn + \tfrac{2ki}{f(k)}) \\ &= 2(Leg1(n)). \end{aligned} \tag{3.8}$$

Thus we have seen that Theorem 2 holds for even k.

Case 2: Let k be an odd positive integer. Then our generating formula for S_k simplifies as follows:

$$(\frac{k}{f(k)}(2m+1),\ \frac{2k}{f(k)^2}(m - \frac{f(k)-1}{2})(m + \frac{f(k)+1}{2}),\ \frac{2k}{f(k)^2}(m - \frac{f(k)-1}{2})(m + \frac{f(k)+1}{2}) + k).$$

Thus as m ranges over the positive integers greater than $\lfloor \frac{f(k)}{2} \rfloor$, every triple of difference k is produced. Once again, we partition the positive integers greater than $\lfloor \frac{f(k)}{2} \rfloor$ into their residue classes mod $f(k)$. We must ensure that only those integers greater than $\lfloor \frac{f(k)}{2} \rfloor$ are produced, namely, we must have:

$$\begin{aligned}
&f(k)n + i > \lfloor \tfrac{f(k)}{2} \rfloor \\
&f(k)n + i > \tfrac{f(k)-1}{2} \\
&2f(k)n + 2i > f(k) - 1 \\
&2f(k)n > f(k) - 1 - 2i \\
&n > \tfrac{1}{2} - \tfrac{1}{2f(k)} - \tfrac{i}{f(k)}.
\end{aligned} \tag{3.9}$$

Thus $f(k)n+i$, as n ranges over the integers greater than $\frac{1}{2} - \frac{1}{2f(k)} - \frac{i}{f(k)}$, produces the ith residue class mod $f(k)$ of the integers greater than $\lfloor \frac{f(k)}{2} \rfloor$. Therefore, substituting $f(k)n + i$ for m in the Theorem 1 formula and allowing n to range over the integers greater than $\frac{1}{2} - \frac{1}{2f(k)} - \frac{i}{f(k)}$ produces the triples generated by the ith residue class. Making this substitution and simplifying, we get:

$$(2kn+\frac{2ik}{f(k)}+\frac{k}{f(k)},\ 2kn^2+\frac{4ki}{f(k)}n+\frac{2k}{f(k)}n+\frac{2ki^2}{f(k)^2}+\frac{2ki}{f(k)^2}-\frac{k}{2}+\frac{1}{4},\ 2kn^2+\frac{4ki}{f(k)}n+\frac{2k}{f(k)}n+\frac{2ki^2}{f(k)^2}+\frac{2ki}{f(k)^2}+\frac{k}{2}+\frac{1}{4}) \tag{3.10}$$

as the generating fomula for the triples produced by the ith residue class. Thus, once again, since every triple produced by the Theorem 1 formula is unique and the residue classes are clearly disjoint, the subsets produced by the residue classes are disjoint. If we take the union of these subsets, we will get every triple generated by the Theorem 1 formula, that is, every triple of a difference k. Thus we can rewrite the set of all triples of difference k as:

$$\begin{aligned}
S_k = \bigcup_{i=0}^{f(k)-1}\{&(2kn + \tfrac{2ik}{f(k)} + \tfrac{k}{f(k)}, 2kn^2 + \tfrac{4ki}{f(k)}n + \tfrac{2k}{f(k)}n + \tfrac{2ki^2}{f(k)^2} + \tfrac{2ki}{f(k)^2} - \tfrac{k}{2} + \tfrac{1}{4}, \\
&2kn^2 + \tfrac{4ki}{f(k)}n + \tfrac{2k}{f(k)}n + \tfrac{2ki^2}{f(k)^2} + \tfrac{2ki}{f(k)^2} - \tfrac{k}{2} + \tfrac{1}{4} + k) \ : n \in Z, n > \tfrac{1}{2} - \tfrac{1}{2f(k)} - \tfrac{i}{f(k)}\}.
\end{aligned} \tag{3.11}$$

Once again allowing $Leg1(n) = 2kn+\frac{2ik}{f(k)}+\frac{k}{f(k)}$ and $Leg2(n) = 2kn^2+\frac{4ki}{f(k)}n+\frac{2k}{f(k)}n+\frac{2ki^2}{f(k)^2}+\frac{2ki}{f(k)^2}-\frac{k}{2}+\frac{1}{4}$ to be the generating formulas for the legs of the triples produced by the ith residue class, we find that:

$$\begin{aligned}
\tfrac{d}{dn}Leg2(n) &= \tfrac{d}{dn}(2kn^2 + \tfrac{4ki}{f(k)}n + \tfrac{2k}{f(k)}n + \tfrac{2ki^2}{f(k)^2} + \tfrac{2ki}{f(k)^2} - \tfrac{k}{2} + \tfrac{1}{4}) \\
&= (4kn + \tfrac{4ki}{f(k)} + \tfrac{2k}{f(k)}) \\
&= 2(2kn + \tfrac{2ki}{f(k)} + \tfrac{k}{f(k)}) \\
&= 2(Leg1(n)).
\end{aligned} \tag{3.12}$$

Thus for k odd we can very similarly write S_k as the union of $f(k)$ disjoint, infinite subsets where the property $\frac{d}{dn}Leg2(n) = 2(Leg1(n))$ holds for the generating formulas of the legs of each subset. □

It is interesting to note that this derivative property in general does not hold for the generating formulas of the legs of the larger set S_k.

4 Conclusion

In summary, we have found and proved two separate but connected formulas that will generate every Pythagorean triple of a fixed difference k. First of all, we found that when when we define

$$f(k) = 2^{\lfloor \frac{n_0}{2} \rfloor} p_1^{\lfloor \frac{n_1}{2} \rfloor} p_2^{\lfloor \frac{n_2}{2} \rfloor} ... p_r^{\lfloor \frac{n_r}{2} \rfloor}, \tag{4.1}$$

where $2^{n_0}p_1^{n_1}p_2^{n_2}...p_r^{n_r}$ is the prime factorization of $2k$, then every triple of a difference k is given by:

$$\left(\tfrac{k}{f(k)}(2m + k \bmod 2),\quad \tfrac{2k}{f(k)^2}(m - \lfloor \tfrac{f(k)}{2} \rfloor)(m + \lceil \tfrac{f(k)}{2} \rceil),\quad \tfrac{2k}{f(k)^2}(m - \lfloor \tfrac{f(k)}{2} \rfloor)(m + \lceil \tfrac{f(k)}{2} \rceil) + k\right)$$

as m runs through the positive integers greater than $\lfloor \frac{f(k)}{2} \rfloor$. To prove this result, we let (x, y, z) be an arbitrary Pythagorean triple with a difference of k and examined both the k even and k odd cases. For k even, we had two subcases: $z - x = k$ and $z - y = k$. For k odd, we had only $z - x = k$. We showed that in every case the triple would be generated by our formula. We also showed that everything generated by the above formula is indeed a Pythagorean triple with difference k. Having completed the proof of our main theorem, we then used the theorem to find an alternate formula that also generates every triple of difference k. Namely, we found that, for even k, every triple of a difference k is given by the following:

$$\bigcup_{i=0}^{f(k)-1} \{(2kn + \frac{2ki}{f(k)},\ 2kn^2 + \frac{4ki}{f(k)}n + \frac{2ki^2}{f(k)^2} - \frac{k}{2},\ 2kn^2 + \frac{4ki}{f(k)}n + \frac{2ki^2}{f(k)^2} + \frac{k}{2})\ : n \in Z, n > \frac{1}{2} - \frac{i}{f(k)}\}, \quad (4.2)$$

and that for odd k, every triple of difference k is given by:

$$\begin{aligned} &\textstyle\bigcup_{i=0}^{f(k)-1} \{(2kn + \frac{2ik}{f(k)} + \frac{k}{f(k)},\ 2kn^2 + \frac{4ki}{f(k)}n + \frac{2k}{f(k)}n + \frac{2ki^2}{f(k)^2} + \frac{2ki}{f(k)^2} - \frac{k}{2} + \frac{1}{4}, \\ &\qquad 2kn^2 + \frac{4ki}{f(k)}n + \frac{2k}{f(k)}n + \frac{2ki^2}{f(k)^2} + \frac{2ki}{f(k)^2} + \frac{k}{2} + \frac{1}{4})\ : n \in Z, n > \frac{1}{2} - \frac{1}{2f(k)} - \frac{i}{f(k)}\}. \end{aligned} \quad (4.3)$$

For each subset in the above unions we noted that the following property holds for the generating formulas of the legs of the triples:

$$\frac{d}{dn}Leg2(n) = 2(Leg1(n)). \quad (4.4)$$

A better understanding of why the above property holds for the given subsets but not for the entire set of Pythagorean triples of difference k, as well as an investigation into other unions that yield the entire set, are areas of future research. Also, the formula in Theorem 1 appears to produce the negative Pythagorean triples. The proof of the theorem, however, made use of Euclid's classification, which only holds for positive triples. Thus another area of future research is investigating whether the formula given in Theorem 1 and its proof can be extended to include all negative Pythagorean triples.

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