1.1 Systems of Linear Equations

Motivation: Consider the system of m equations and n unknowns

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad\qquad\qquad &\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

Question 1: Does a solution exist?

Question 2: How many solutions exist?

Question 3: How ~~we~~ do we find all the solutions?

Ex: The system of 2 equations and 3 unknowns

$$\begin{aligned}
x + y + z &= 2 \\
x - 2z &= 3
\end{aligned}$$

is solved by both $(x,y,z) = (5,-4,1)$ and $(x,y,z) = (1,2,-1)$.
(The system has more than one solution)

Ex: The system of 3 equations and 2 unknowns

$$\begin{aligned}
x + 3y &= 2 \quad (1) \\
-2x + 7y &= 8 \quad (2) \\
4x - 14y &= 0 \quad (3)
\end{aligned}$$

has no solutions. $2(2) + (3) = 0$ on left but 16 on right.

Ex: Find all solutions to the system

$$x + 2y = 3 \quad (1)$$

$$3x + 4y = 0 \quad (2)$$

Bad Idea: Start deriving new equations:

Equation (2) $\Rightarrow$

$$y = -\frac{3}{4}x \quad (3)$$

Plug (3) into (1) to obtain

$$x + 2\left(-\frac{3}{4}x\right) = 3$$

$$x = -6 \quad (4)$$

Plug (4) into (3) to obtain

$$y = -\frac{3}{4}(-6) = \frac{9}{2} \quad (5)$$

Plug (4) and (5) into (1) and (2) to see that the solution is $(x,y) = \left(-6, \frac{9}{2}\right)$. ▨

Why is this a bad idea?

1. Equations get messy in high dimensions

2. The procedure is not guarenteed to terminate.

<u>Def</u>: Two systems are <u>equivalent</u> if they have the same solutions.

<u>Good Idea</u>: To solve a given system, replace it with an equivalent "easy" system.

Def: The augmented matrix of the system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\quad \vdots \qquad\quad \ddots \qquad \vdots \quad &\quad\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

is

$$\left[\begin{array}{cccc:c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right] = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix}$$

Ex: The augmented matrix of

$$\begin{aligned} 3x + 4y - 8z &= 0 \\ y + 10z &= 6 \\ 9x + 7y - 2z &= -5 \end{aligned}$$

is

$$\left[\begin{array}{ccc:c} 3 & 4 & -8 & 0 \\ 0 & 1 & 10 & 6 \\ 9 & 7 & -2 & -5 \end{array}\right]$$

Def: An augmented matrix is in reduced row-echelon form (rref) if

1. any zero-rows appear at the bottom
2. the first nonzero entry of a nonzero row is 1 (called a leading 1)
3. the leading 1 of a nonzero row appears to the right of the leading 1 of any proceeding row
4. all the other entries of a column containing a leading 1 are zero

Ex: The following augmented matrices are rref

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 2 \end{array}\right] \quad \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \left[\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Ex: The following augmented matrices are not rref

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \quad \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Def: A column in a rref augmented matrix corresponds to a

1. free variable if it does not contain a leading 1
2. dependent variable if it contains a leading 1

Def: A rref system is consistent if it does not have a leading 1 in its last column.

A rref system is incosistent if it has a leading 1 in its last column.

Note: Solving rref systems is easy: identify the free variables and then write the dependent variables in terms of the free variables.

Ex: The rref system

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad (\text{consistent})$$

has free variables x_2, x_4 and dependent variables x_1, x_3. The solutions are then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ x_2 \\ -1-2x_4 \\ x_4 \end{bmatrix}$$

Ex: The rref system

$$\left[\begin{array}{ccc:c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 8 \end{array}\right]$$

~~(Inco~~ (consistent)

has no free variables. The solution is then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 8 \end{bmatrix}$$

Ex: The rref system

$$\left[\begin{array}{cccc:c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

(inconsistent)

has free variables x_1, x_3 and has dependent variables x_2, x_4. However, the last row corresponds to the equation $0 = 1$, which is impossible! This system has no solutions.

Thm: A rref system

1. Has a unique solution if it is consistent and has no free variables
2. Has infinitely many solutions if it is consistent and has free variables
3. Has no solutions if it is inconsistent.

Def: There are three types of elementary row operations on an augmented matrix. They are

Row Switching: A row within the augmented matrix can be switched with another row. $R_i \longleftrightarrow R_j$

Row Multiplication: Each row can be multiplied by a non zero constant.

$$\lambda R_i \longrightarrow R_i \quad \text{where } \lambda \neq 0$$

Row Addition: A row can be replaced by the sum of that row and a multiple of another row. $R_i + \lambda R_j \longrightarrow R_i$ where $i \neq j$.

Ex: (Row Switching)

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 2 & 4 \\ 8 & -1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 3 & 5 \\ 0 & -2 & -3 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \longleftrightarrow R_4} \left[\begin{array}{cccc|c} 1 & 0 & 3 & 2 & 4 \\ 0 & -2 & -3 & 0 & 1 \\ 2 & 1 & 0 & 3 & 5 \\ 8 & -1 & 0 & 0 & 1 \end{array}\right]$$

Ex: (Row Multiplication)

$$\left[\begin{array}{ccc|c} -2 & 0 & 1 & 9 \\ 3 & 3 & 4 & 2 \\ 0 & 1 & 3 & 6 \end{array}\right] \xrightarrow{-3R_3 \longrightarrow R_3} \left[\begin{array}{ccc|c} -2 & 0 & 1 & 9 \\ 3 & 3 & 4 & 2 \\ 0 & -3 & -9 & -18 \end{array}\right]$$

Ex: (Row Addition)

$$\left[\begin{array}{ccc|c} 8 & 3 & 1 & 2 \\ 0 & 4 & 9 & 7 \\ 6 & 8 & 2 & 4 \end{array}\right] \xrightarrow{R_3 - \frac{1}{3}R_1 \longrightarrow R_3} \left[\begin{array}{ccc|c} 8 & 3 & 1 & 2 \\ 0 & 4 & 9 & 7 \\ \frac{10}{3} & 7 & \frac{5}{3} & \frac{10}{3} \end{array}\right]$$

Thm: Performing elementary row operations on an augmented matrix yields an equivalent system.

Idea: To solve a given system, use elementary row operations to turn it into a rref system.

Thm: (Gauß-Jordan Elimination) Every augmented matrix is equivalent to a unique rref augmented matrix. Furthermore, its rref may be obtained via elementary row operations.

Ex: Find all solutions to the system

$$x + 2y = 3$$
$$3x + 4y = 0.$$

Solution: Row reduce

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 3 & 4 & 0 \end{array}\right] \xrightarrow{R_2 - 3R_1 \to R_2} \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -2 & -9 \end{array}\right]$$

$$\xrightarrow{R_1 + R_2 \to R_1} \left[\begin{array}{cc|c} 1 & 0 & -6 \\ 0 & -2 & -9 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{2} R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 0 & -6 \\ 0 & 1 & 9/2 \end{array}\right]$$

$$\Rightarrow \operatorname{rref} \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 3 & 4 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & -6 \\ 0 & 1 & 9/2 \end{array}\right].$$

The rref of the system is consistant and has no free variables.
Hence the unique solution is $(x, y) = \left(-6, \frac{9}{2}\right)$. ∎

Ex: Find all solutions to the system

$$4x_2 + x_3 = 2$$

$$2x_1 + 6x_2 - 2x_3 = 3$$

$$4x_1 + 8x_2 - 5x_3 = 4$$

Solution: Row reduce:

$$\left[\begin{array}{ccc|c} 0 & 4 & 1 & 2 \\ 2 & 6 & -2 & 3 \\ 4 & 8 & -5 & 4 \end{array}\right] \xrightarrow{R_1 \longleftrightarrow R_2} \left[\begin{array}{ccc|c} 2 & 6 & -2 & 3 \\ 0 & 4 & 1 & 2 \\ 4 & 8 & -5 & 4 \end{array}\right]$$

$$\xrightarrow{R_3 - 2R_1 \longrightarrow R_3} \left[\begin{array}{ccc|c} 2 & 6 & -2 & 3 \\ 0 & 4 & 1 & 2 \\ 0 & -4 & -1 & -2 \end{array}\right]$$

$$\xrightarrow{R_3 + R_2 \longrightarrow R_3} \left[\begin{array}{ccc|c} 2 & 6 & -2 & 3 \\ 0 & 4 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{4}R_2 \longrightarrow R_2} \left[\begin{array}{ccc|c} 2 & 6 & -2 & 3 \\ 0 & 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 - 6R_2} \left[\begin{array}{ccc|c} 2 & 0 & -7/2 & 0 \\ 0 & 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{7}{4} & 0 \\ 0 & 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

gives

rref consistent

$$\operatorname{rref}\left[\begin{array}{ccc|c} 0 & 4 & 1 & 2 \\ 2 & 6 & -2 & 3 \\ 4 & 8 & -5 & 5 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & -\frac{7}{4} & 0 \\ 0 & 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The rref of the system is consistent and has free variable x_3. Hence the system has infinitely many solutions. The solutions are given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{7}{4}x_3 \\ \frac{1}{2} - \frac{1}{4}x_3 \\ x_3 \end{bmatrix}. \quad \blacksquare$$

Def: A homogeneous system is a system of the form

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= 0 \\
\vdots \qquad\quad \vdots \qquad\quad \ddots \quad\ \ \vdots \quad &\ \ \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= 0 .
\end{aligned}
$$

The solution $x_1 = x_2 = \dots = x_n = 0$ is the trivial solution.
All other solutions are called nontrivial solutions.

Thm: The homogeneous system of m equations in n variables has infinitely many solutions when $m < n$.