1.2 Matrices and Matrix Operations

Def: A matrix of dimension $m \times n$ is an object of the form

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}]$$

where $a_{ij} \in \mathbb{R}$.

Ex: The following are matrices

$$A = [a_{ij}] = \begin{bmatrix} 2 & 0 \\ 4 & 1 \\ 16 & -3/4 \end{bmatrix} \qquad \dim A = 3 \times 2, \quad a_{21} = 4$$

$$B = [b_{ij}] = \begin{bmatrix} 3 & 0 & 8 & 1 & 2 \\ 7 & -\frac{1}{2} & 3 & 2 & 10 \\ 32 & 1 & 0 & 0 & 0 \end{bmatrix} \qquad \dim B = 3 \times 5, \quad b_{32} = 1.$$

Notation: We will often denote the entries of a matrix $A = [a_{ij}]$ by

$$a_{ij} = [A]_{ij} = \operatorname{ent}_{ij}(A).$$

Def: The collection of all matrices of dimension $m \times n$ will be denoted by $M_{m \times n}(\mathbb{R})$.

Def: Matrices in $M_{1 \times n}(\mathbb{R})$ are called row vectors. Matrices in $M_{m \times 1}(\mathbb{R})$ are called column vectors.

Ex: We have the identifications

$$M_{2\times 2}(\mathbb{R}) = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} : a_{ij} \in \mathbb{R} \right\}$$

$$\mathbb{R}^n = M_{n\times 1}(\mathbb{R}) = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} : a_i \in \mathbb{R} \right\}.$$

Def: Let $A, B \in M_{m\times n}(\mathbb{R})$. The sum of A and B is the $m\times n$ matrix whose entries are

$$[A+B]_{ij} = a_{ij} + b_{ij}.$$

Ex: We have

$$\begin{bmatrix} 3 & 1 & 8 \\ 2 & 4 & -1 \end{bmatrix} + \begin{bmatrix} 0 & -3 & -4 \\ 1 & 2 & -7 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 4 \\ 3 & 6 & -8 \end{bmatrix}.$$

Note: $A+B$ is only defined when A and B have the same dimension.

Def: The scalar product of $\lambda \in \mathbb{R}$ and $A \in M_{m\times n}(\mathbb{R})$ is the $m\times n$ matrix whose entries are

$$\cancel{\lambda AB} \quad [\lambda A]_{ij} = \lambda a_{ij}.$$

Ex: We have

$$6 \begin{bmatrix} 1 & -2 \\ 3 & 1 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 6 & -12 \\ 18 & 6 \\ 0 & 24 \end{bmatrix}$$

Thm: Let $A, B, C \in M_{m \times n}(\mathbb{R})$ and let $\alpha, \beta \in \mathbb{R}$. Then

① $A + B = B + A$

② $A + (B + C) = (A + B) + C$

③ $\alpha(\beta A) = (\alpha\beta) A$

④ $\alpha(A + B) = \alpha A + \alpha B$

⑤ $(\alpha + \beta) A = \alpha A = \beta A.$

pf: ① Note that

$$[A + B]_{ij} = a_{ij} + b_{ij} = b_{ij} + a_{ij} = [B + A]_{ij}$$

so that $A + B = B + A$.

② Note that

$$[A + (B + C)]_{ij} = a_{ij} + (b_{ij} + c_{ij}) = (a_{ij} + b_{ij}) + c_{ij} = [(A + B) + C]_{ij}$$

so that $A + (B + C) = (A + B) + C$.

③ Note that

$$[\alpha(\beta A)]_{ij} = \alpha [\beta A]_{ij} = \alpha(\beta a_{ij}) = (\alpha\beta) a_{ij} = [(\alpha\beta) A]_{ij}$$

So that $\alpha(\beta A) = (\alpha\beta) A$.

④ Note that

$$\left[\alpha\,(A+B)\right]_{ij} = \alpha\,[A+B]_{ij} = \alpha\,(a_{ij}+b_{ij}) = \alpha a_{ij} + \alpha b_{ij}$$

$$= [\alpha A]_{ij} + [\alpha B]_{ij}$$

So that $\alpha(A+B) = \alpha A + \alpha B$.

⑤ Note that

$$\left[(\alpha+\beta)\,A\right]_{ij} = (\alpha+\beta)\,a_{ij} = \alpha a_{ij} + \beta a_{ij} = [\alpha A]_{ij} + [\beta A]_{ij}$$

So that $(\alpha+\beta)A = \alpha A + \beta A$. ∎

Def: The zero matrix of dimension $m \times n$ is the matrix $0_{m\times n} \in M_{m\times n}(\mathbb{R})$ whose entries are given by $[0_{m\times n}]_{ij} = 0$.

Thm: Let $A \in M_{m\times n}(\mathbb{R})$. Then

① $A + 0_{m\times n} = A$

② $0 \cdot A = 0_{m\times n}$

③ $A - A = 0_{m\times n}$.

pf: Exercise. ∎

Def: Let $A \in M_{\ell \times m}(\mathbb{R})$ and let $B \in M_{m \times n}(\mathbb{R})$. The product of A and B is the matrix $AB \in M_{\ell \times n}(\mathbb{R})$ whose entries are

$$[AB]_{ij} = \sum_{k=1}^{m} [A]_{ik} [B]_{kj} = \sum_{k=1}^{m} a_{ik} b_{kj}.$$

Note: The product AB is only defined when

# of columns of A = # of rows of B.

Ex: Find AB where

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 2 & 0 \end{bmatrix}.$$

Solution: Note that $\dim(A) = 2 \times 3$ and $\dim(B) = 3 \times 2$. Therefore AB is defined and $\dim(AB) = 2 \times 2$. The entries of AB are

$$[AB]_{11} = \sum_{k=1}^{3} a_{1k} b_{k1}$$
$$= a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}$$
$$= (1)(0) + (0)(1) + (-1)(2)$$
$$= -2$$

$$[AB]_{12} = \sum_{k=1}^{3} a_{1k} b_{k2}$$
$$= a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32}$$
$$= (1)(-1) + (0)(0) + (-1)(0)$$
$$= -1$$

$$[AB]_{21} = \sum_{k=1}^{3} a_{2k} b_{k1}$$
$$= a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31}$$
$$= (0)(0) + (1)(1) + (2)(2)$$
$$= 5$$

$$[AB]_{22} = \sum_{k=1}^{3} a_{2k} b_{k2}$$
$$= a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32}$$
$$= (0)(-1) + (1)(0) + (2)(0)$$
$$= 0$$

So that

$$AB = \begin{bmatrix} -2 & -1 \\ 5 & 0 \end{bmatrix}. \quad \blacksquare$$

Note: The formula for AB works well for proofs, but for computing AB by hand it is easier to think in terms of rows and columns:

$$[AB]_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = \begin{bmatrix} & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ & & & \end{bmatrix} \begin{bmatrix} & b_{1j} & \\ & b_{2j} & \\ & \vdots & \\ & b_{nj} & \end{bmatrix}$$

Ex: With this perspective,

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 5 & 0 \end{bmatrix}.$$

Ex: Find AB where

$$A = \begin{bmatrix} 3 & 4 \\ 1 & -8 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 3 & 1 \end{bmatrix}.$$

Solution: Note that $\dim(A) = 2 \times 2$ while $\dim(B) = 3 \times 2$. Since $2 \neq 3$, AB is not defined. $\blacksquare$

Thm: Given matrices A, B, and C, let $\lambda \in \mathbb{R}$. Then

1. $A(BC) = (AB)C$
2. $A(B+C) = AB + AC$
3. $(A+B)C = AC + BC$
4. $\lambda(AB) = (\lambda A)B = A(\lambda B)$

whenever the indicated sums and products are defined.

pf: 1. Here, $A \in M_{k \times l}(\mathbb{R})$, $B \in M_{l \times m}(\mathbb{R})$, and $C \in M_{m \times n}(\mathbb{R})$. Then

$$[A(BC)]_{ij} = \sum_{p=1}^{l} a_{ip} [BC]_{pj} = \sum_{p=1}^{l} a_{ip} \sum_{q=1}^{m} b_{pq} c_{qj}$$

$$= \sum_{p=1}^{l} \sum_{q=1}^{m} a_{ip} (b_{pq} c_{qj}) = \sum_{q=1}^{m} \sum_{p=1}^{l} (a_{ip} b_{pq}) c_{qj}$$

$$= \sum_{q=1}^{m} [AB]_{iq} c_{qj} = [(AB)C]_{ij}$$

So that $A(BC) = (AB)C$.

② Here, $A \in M_{\ell \times m}(\mathbb{R})$ and $B, C \in M_{m \times n}(\mathbb{R})$. Then

$$[A(B+C)]_{ij} = \sum_{k=1}^{m} a_{ik}[B+C]_{kj} = \sum_{k=1}^{m} a_{ik}(b_{kj} + c_{kj})$$

$$= \sum_{k=1}^{m} (a_{ik}b_{kj} + a_{ik}c_{kj}) = \sum_{k=1}^{m} a_{ik}b_{kj} + \sum_{k=1}^{m} a_{ik}c_{kj}$$

$$= [AB]_{ij} + [AC]_{ij}$$

$$= [AB + AC]_{ij}$$

So that $A(B+C) = AB + AC$.

③, ④: Exercise. $\blacksquare$

Def: The identity matrix of dimension $n \times n$ is the matrix $I_n \in M_{n\times n}(\mathbb{R})$ whose entries are

$$[I_n]_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} = \delta_{ij}.$$

Ex: We have

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thm: Let $A \in M_{m\times n}(\mathbb{R})$. Then

① $O_{\ell\times m} A = O_{\ell n}$ and $A\, O_{n\times \ell} = O_{m\times \ell}$

② $I_m A = A I_n = A$.

pf: ① For the first equality, note that

$$[O_{\ell\times m} A]_{ij} = \sum_{k=1}^{m} [O_{\ell\times m}]_{ik} a_{kj} = \sum_{k=1}^{m} 0\, a_{kj} = 0 = [O_{\ell\times m}]_{ij}$$

so that $O_{\ell\times m} A = O_{\ell\times m}$. The second equality is similar.

② For the first equality, note that

$$[I_m A]_{ij} = \sum_{k=1}^{m} [I_m]_{ik} a_{kj} = \sum_{k=1}^{m} \delta_{ik} a_{kj} = a_{ij}$$

$$= \sum_{k=1}^{n} a_{ik} \delta_{kj} = \sum_{k=1}^{n} a_{ik} [I_n]_{kj} = [A I_n]_{ij}$$

so that $I_m A = A I_n$. The second equality is similar. ∎

Note: The previous two theorems show that matrix arithmetic is similar to usual arithmetic. However, there are some key differences.

Ex: Find AB and BA where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Solution: Compute

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

So, $AB \neq BA$. Furthermore, $AB = 0_{2\times 2}$ even though $A \neq 0_{2\times 2}$, $B \neq 0_{2\times 2}$. ■

Motivation: We can write the linear system

$$
\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\
\vdots & & \vdots & & \ddots & & \vdots & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m
\end{array}
$$

may be written as

$$AX = B$$

where

$$
A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & \cancel{a_{mm}}\ a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.
$$

Ex: The system ~~corresponds to~~

$$
\begin{aligned}
2x - y + 4z &= 1 \\
x - 7y + z &= 3 \\
-x + 2y + z &= 2
\end{aligned}
$$

corresponds to

$$
\begin{bmatrix} 2 & -1 & 4 \\ 1 & -7 & 1 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}.
$$