1.3 Inverses of Matrices

Def: Let $A \in M_{n\times n}(\mathbb{R})$. An inverse of A is a matrix $B \in M_{n\times n}(\mathbb{R})$ such that $AB = BA = I$.

Ex: Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$BA = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

Hence B is an inverse of A. ∎

Ex: The matrix $O_{n\times n}$ does not have an inverse since $O_{n\times n}B = O_{n\times n}$ whenever $B \in M_{n\times n}(\mathbb{R})$.

Ex: The matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

has no inverse since

$$AB = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} b_{11}+b_{21} & b_{12}+b_{22} \\ 0 & 0 \end{bmatrix} \neq I$$

whenever $B \in M_{2\times 2}(\mathbb{R})$. ∎

Def: A matrix is invertible or nonsingular if it has an inverse. A matrix is noninvertible or singular if it does not have an inverse.

Def: The collection of all $n \times n$ invertible matrices is denoted by $GL_n(\mathbb{R})$.

Thm: (Inverse are unique) Let $A \in GL_n(\mathbb{R})$ have inverses B_1 and B_2. Then $B_1 = B_2$.

pf: $B_1 = B_1 I_n = B_1(AB_2) = (B_1 A) B_2 = I_n B_2 = B_2$. ■

Note: Since inverses are unique, we may, when convenient, denote the inverse of A by A^{-1}!

Question: How do we find A^{-1}?

Answer:

① Form the augmented matrix $[A \,\vdots\, I_n]$

② Use elementary row operations to reduce the left-hand portion into reduced row-echelon form $\rightsquigarrow$ $[\text{rref}(A) \,\vdots\, B]$

③ If $\text{rref}(A) = I$, then $B = A^{-1}$. Otherwise, A is not invertible.

Ex: Find the inverse of

$$\begin{bmatrix} 2 & 1 & 3 \\ 2 & 1 & 1 \\ 4 & 5 & 1 \end{bmatrix}$$

Solution: Row reduce

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 4 & 5 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 - 2R_1 \to R_3]{R_2 - R_1 \to R_2} \left[\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 0 & 0 \\ 0 & 0 & -2 & -1 & 1 & 0 \\ 0 & 3 & -5 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 0 & 0 \\ 0 & 3 & -5 & -2 & 0 & 1 \\ 0 & 0 & -2 & -1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[R_2 - \frac{5}{2} R_3 \to R_2]{R_1 + \frac{3}{2} R_3 \to R_1} \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & -\frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 3 & 0 & \frac{1}{2} & -\frac{5}{2} & 1 \\ 0 & 0 & -2 & -1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 - \frac{1}{3} R_2 \to R_1} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & -\frac{2}{3} & \frac{7}{3} & -\frac{1}{3} \\ 0 & 3 & 0 & \frac{1}{2} & -5/2 & 1 \\ 0 & 0 & -2 & -1 & 1 & 0 \end{array}\right]$$

$$\begin{array}{l} \frac{1}{2}R_1 \to R_1 \\ \frac{1}{3}R_2 \to R_2 \\ -\frac{1}{2}R_3 \to R_3 \end{array} \longrightarrow \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -1/3 & 7/6 & -1/6 \\ 0 & 1 & 0 & 1/6 & -5/6 & 1/3 \\ 0 & 0 & 1 & 1/2 & -1/2 & 0 \end{array}\right].$$

This gives

$$A^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{7}{6} & -\frac{1}{6} \\ \frac{1}{6} & -\frac{5}{6} & \frac{1}{3} \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}. \quad \blacksquare$$

Def: An elementary matrix of dimension $n \times n$ is a matrix that results by performing an elementary row operation on I_n.

Ex: The following are elementary matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{3R_2 \to R_2} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 - \frac{1}{2}R_1 \to R_2} \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Idea: ~~I~~ Multiplication by elementary matrices on the left of a matrix ~~corre~~ gives the corresponding elementary row operation.

Ex: We have

$$\begin{bmatrix} 1 & 7 \\ 0 & \frac{1}{3} \end{bmatrix} \xrightarrow{3R_2 \to R_2} \begin{bmatrix} 1 & 7 \\ 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 7 \\ 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 7 \\ 0 & 1 \end{bmatrix}.$$

Thm: Elementary matrices are invertible and the inverses are

$$\left[R_i \leftrightarrow R_j\right]^{-1} = \left[R_i \leftrightarrow R_j\right]$$

$$\left[\lambda R_i \to R_i\right]^{-1} = \left[\frac{1}{\lambda} R_i \to R_i\right]$$

$$\left[R_i + \lambda R_j \to R_i\right]^{-1} = \left[R_i - \lambda R_j \to R_i\right].$$

<u>Ex</u>: Write A^{-1} as a product of elementary matrices where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}.$$

<u>Solution</u>: Row reduce

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 5 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - 3R_1 \to R_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -3 & 1 \end{array}\right]$$

$$\xrightarrow{-R_2 \to R_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 3 & -1 \end{array}\right]$$

$$\xrightarrow{R_1 - 2R_2 \to R_1} \left[\begin{array}{cc|cc} 1 & 0 & -5 & 2 \\ 0 & 1 & 3 & -1 \end{array}\right].$$

This gives

$$A^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}.$$

These row reductions correspond to elementary matrices

$$R_2 - 3R_1 \to R_2 \rightsquigarrow \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} = E_1$$

$$-R_2 \to R_2 \rightsquigarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = E_2$$

$$R_1 - 2R_2 \to R_1 \rightsquigarrow \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = E_3 .$$

This gives

$$A^{-1} = E_3 E_2 E_1 . \quad \blacksquare$$

Thm: Given $A, B \in Gl_n(\mathbb{R})$, let $\lambda \in \mathbb{R}$ such that $\lambda \neq 0$. Then

1. $A^{-1} \in Gl_n(\mathbb{R})$ and $(A^{-1})^{-1} = A$
2. $\lambda A \in Gl_n(\mathbb{R})$ and $(\lambda A)^{-1} = \frac{1}{\lambda} A^{-1}$
3. $AB \in Gl_n(\mathbb{R})$ and $(AB)^{-1} = B^{-1}A^{-1}$
4. $A^k \in Gl_n(\mathbb{R})$ and $(A^k)^{-1} = (A^{-1})^k$.

pf: Exercise. ∎

Thm: (Fundamental Theorem of Invertible Matrices) Let $A \in M_{n\times n}(\mathbb{R})$.
Then the following are equivalent:

1. $A \in Gl_n(\mathbb{R})$
2. $AX = B$ has a unique solution for every $B \in \mathbb{R}^n$
3. $AX = 0$ has only the trivial solution
4. $\mathrm{rref}(A) = I_n$
5. A is a product of elementary matrices.

pf: Exercise. ∎

Thm: Let $A, B \in M_{n\times n}(\mathbb{R})$ such that $AB = I_n$ or $BA = I_n$. Then $B = A^{-1}$.

pf: Suppose $BA = I_n$ and consider the system $AX = 0$. Then

$$0 = B0 = B(AX) = (BA)X = IX = X.$$

So, the system $AX = 0$ only has the trivial solution. By the fundamental theorem, $A \in GL_n(\mathbb{R})$ and

$$B = BI = B(AA^{-1}) = (BA)A^{-1} = IA^{-1} = A^{-1}.$$

The argument when $AB = I$ is an exercise. ∎