1.4 Special Matrices & Additional Properties

Def: A matrix $A \in M_{n\times n}(\mathbb{R})$ is diagonal if $a_{ij} = 0$ for $i \neq j$.

Ex: The matrices

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_n, \quad O_{n\times n}$$

are diagonal.

Note: We write

$$\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

Thm: Let

$$A = \mathrm{diag}(a_1, a_2, \ldots, a_n)$$
$$B = \mathrm{diag}(b_1, b_2, \ldots, b_n).$$

Then

1. $A + B = \mathrm{diag}(a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$
2. $AB = \mathrm{diag}(a_1 b_1, a_2 b_2, \ldots, a_n b_n)$
3. $A \in GL_n(\mathbb{R})$ if and only if each $a_i \neq 0$. Furthermore, in this case, $A^{-1} = \mathrm{diag}\left(\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n}\right)$.

Def: A matrix $A \in M_{n\times n}(\mathbb{R})$ is

1. **upper triangular** if $a_{ij} = 0$ for $j > i$
2. **lower triangular** if $a_{ij} = 0$ for $i > j$.

Ex: We have

$$\begin{bmatrix} 1 & 4 & 2 \\ 0 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \text{upper triangular}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 8 & 0 \\ 4 & 9 & 7 \end{bmatrix} \longrightarrow \text{lower triangular}$$

Thm: Let $A, B \in M_{n\times n}(\mathbb{R})$ be upper triangular. Then

1. $A+B$ is upper triangular
2. AB is upper triangular
3. A is invertible if and only if ~~the di~~ each of the diagonal entries of A is nonzero.

The ~~similar~~ corresponding theorem also holds for lower triangular matrices.

Def: Let $A \in M_{m\times n}(\mathbb{R})$. The transpose of A is the $n\times m$ matrix A^T whose entries are

$$[A^T]_{ij} = a_{ji}.$$

Ex: We have

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

Thm: Given matrices A and B, let $\lambda \in \mathbb{R}$. Then

① $(A^T)^T = A$

② $(A+B)^T = A^T + B^T$

③ $(\lambda A)^T = \lambda A^T$

④ $(AB)^T = B^T A^T$

⑤ $(A^T)^{-1} = (A^{-1})^T$.

whenever the indicated sums and products are defined.

pf: Exercise. ∎

Def: A matrix A is symmetric if $A^T = A$.

Ex: We have

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \longrightarrow \text{Symmetric}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \longrightarrow \text{not symmetric.}$$

Thm: Let $A, B \in M_{n \times n}(\mathbb{R})$. Then

1. if A and B are symmetric, then so is $A + B$
2. if A is symmetric, then so is λA for $\lambda \in \mathbb{R}$
3. $A^T A$ and $A A^T$ are symmetric
4. if A is invertible and symmetric, then A^{-1} is symmetric.