# 1.5 Determinants

<u>Def</u>: Let $A \in M_{n \times n}(\mathbb{R})$. The $(i,j)$-<u>minor</u> of $A$ is the $(n-1) \times (n-1)$ matrix obtained by deleting the $i^{\text{th}}$ row and $j^{\text{th}}$ column of $A$. It is denoted by $A_{ij}$ (book uses $M_{ij}$).

<u>Ex</u>: We have

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 0 & 10 \\ 0 & 1 & 6 \end{bmatrix} \Rightarrow A_{21} = \begin{bmatrix} \cancel{1} & 2 & 3 \\ \cancel{7} & \cancel{0} & \cancel{10} \\ \cancel{0} & 1 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 6 \end{bmatrix}$$

<u>Def</u>: The <u>determinant</u> of $A \in M_{1 \times 1}(\mathbb{R})$ is $\det(A) = a_{11}$.

<u>Def</u>: The <u>determinant</u> of $A \in M_{n \times n}(\mathbb{R})$ is

$$\det(A) = \sum_{k=1}^{n} (-1)^{1+k} a_{1k} \det(A_{1k}).$$

<u>Note</u>: We often write

$$\det(A) = |A|.$$

Ex: Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Find $\det(A)$.

Solution: Compute

$$\det(A) = \sum_{k=1}^{2} (-1)^{1+k} a_{1k} \det(A_{1k})$$

$$= (-1)^{1+1} a_{11} \det(A_{11}) + (-1)^{1+2} a_{12} \det(A_{12})$$

$$= a_{11} \det(d) - a_{12} \det(c)$$

$$= ad - bc. \quad \blacksquare$$

Ex: Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Find det(A).

Solution: Compute

$$\det(A) = \sum_{k=1}^{3} (-1)^{1+k} a_{1k} \det(A_{1k})$$

$$= (-1)^{1+1} a_{11} \det(A_{11}) + (-1)^{1+2} a_{12} \det(A_{12}) + (-1)^{1+3} a_{13} \det(A_{13})$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}\left(a_{22}a_{33} - a_{32}a_{23}\right) - a_{12}\left(a_{21}a_{33} - a_{31}a_{23}\right)$$

$$+ a_{13}\left(a_{21}a_{32} - a_{31}a_{22}\right). \blacksquare$$

<u>Def</u>: Let $A \in M_{n \times n}(\mathbb{R})$. The $(i,j)$-<u>cofactor</u> of $A$ is

$$C_{ij} = (-1)^{i+j} \det(A_{ij}).$$

<u>Note</u>: With this notation, our definition of determinants is

$$\det(A) = \sum_{k=1}^{n} a_{1k} C_{1k}.$$

<u>Ex</u>: For

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

we have

$$C_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

Note: The signs of the cofactors can be remembered by

$$\begin{matrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}$$

Thm: (Laplace Cofactor Expansion Theorem). Let $A \in M_{n \times n}(\mathbb{R})$. Then for any $1 \le i \le n$,

$$\det(A) = \sum_{k=1}^{n} a_{ik} C_{ik}$$

$$\det(A) = \sum_{k=1}^{n} a_{ki} C_{ki}.$$

Idea: The Laplace Cofactor Expansion Theorem allows us to compute determinants by "expanding" about any row or column. This is nice because some rows ~~and~~ or columns are easier to expand about than others.

<u>Ex</u>: Compute

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

<u>Solution</u>: We can expand several rows or colums:

<u>Row 2</u>: $$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = (-4)\begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} + (5)\begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} + (-6)\begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix}$$

<u>Column 3</u>:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = (3)\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} + (-6)\begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} + (9)\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}. \blacksquare$$

Ex: Compute

$$\begin{vmatrix} 7 & -3 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 2 & 1 & -2 & -5 \\ 0 & 4 & 0 & 6 \end{vmatrix}$$

Solution: Expansion about column 3 gives

$$\begin{vmatrix} 7 & -3 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 2 & 1 & -2 & -5 \\ 0 & 4 & 0 & 6 \end{vmatrix} = (-2) \begin{vmatrix} 7 & -3 & 4 \\ 0 & 1 & 3 \\ 0 & 4 & 6 \end{vmatrix}.$$

Then expand about column 1 to get

$$\begin{vmatrix} 7 & -3 & 4 \\ 0 & 1 & 3 \\ 0 & 4 & 6 \end{vmatrix} = 7 \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} = 7(6-12) = -42.$$

Hence

$$\begin{vmatrix} 7 & -3 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 2 & 1 & -2 & -5 \\ 0 & 4 & 0 & 6 \end{vmatrix} = (-2)(-42) = 84. \blacksquare$$

Thm: If $A$ has a row or column of zeros, then $\det(A) = 0$.

Thm: The determinant of a triangular matrix is the product of its diagonal entries.

Thm: $\det(A^T) = \det(A)$

Thm: Let $A \in M_{n \times n}(\mathbb{R})$. Then

1. if $B$ is obtained by interchanging two rows of $A$, then $\det(B) = -\det(A)$
2. if $B$ is obtained by multiplying a row of $A$ by $\lambda \in \mathbb{R}$, then $\det(B) = \lambda \det(A)$
3. if $B$ is obtained by replacing a row of $A$ by itself plus a multiple of another row of $A$, then $\det(B) = \det(A)$.

Idea: determinants play well with elementary row operations

1. $A \xrightarrow{R_i \leftrightarrow R_j} B \Rightarrow \det(A) = -\det(B)$
2. $A \xrightarrow{\lambda R_i \to R_i} B \Rightarrow \det(A) = \frac{1}{\lambda}\det(B)$
3. $A \xrightarrow{R_i + \lambda R_j \to R_i} B \Rightarrow \det(A) = \det(B)$

Ex: Find

$$\det \begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & 1 & 2 & 1 \\ 1 & 1 & -1 & -2 \\ 1 & -1 & 1 & 4 \end{bmatrix}.$$

Solution: Row reduce

$$\begin{vmatrix} 1 & -1 & 2 & 3 \\ 2 & 1 & 2 & 1 \\ 1 & 1 & -1 & -2 \\ 1 & -1 & 1 & 4 \end{vmatrix} \quad \begin{array}{l} R_2 - 2R_1 \to R_2 \\ R_3 - R_1 \to R_3 \\ R_4 - R_1 \to R_4 \end{array} \quad \begin{vmatrix} 1 & -1 & 2 & 3 \\ 0 & 3 & -2 & -5 \\ 0 & 2 & -3 & -5 \\ 0 & 0 & -1 & 1 \end{vmatrix}$$

$$\xrightarrow{R_3 - \frac{2}{3}R_2 \to R_3} \begin{vmatrix} 1 & -1 & 2 & 3 \\ 0 & 3 & -2 & -5 \\ 0 & 0 & -\frac{5}{3} & -\frac{5}{3} \\ 0 & 0 & -1 & 1 \end{vmatrix}$$

$$\xrightarrow{R_4 - \frac{3}{5}R_3 \to R_4} \begin{vmatrix} 1 & -1 & 2 & 3 \\ 0 & 3 & -2 & -5 \\ 0 & 0 & -\frac{5}{3} & -\frac{5}{3} \\ 0 & 0 & 0 & 2 \end{vmatrix}$$

gives $\det = (1)(3)\left(\frac{-5}{3}\right)(2) = -10.$ ∎