1.6 Further Properties of Determinants

Recall: The (i,j)-*minor* of $A \in M_{n \times n}(\mathbb{R})$ is the $(n-1)\times(n-1)$ matrix A_{ij} obtained by deleting the i^{th} row and j^{th} column of A.

The (i,j)-*cofactor* of A is the number $(-1)^{i+j}\det(A_{ij})$.

Thm: (Laplace Cofactor Expansion Theorem) Let $A \in M_{n\times n}(\mathbb{R})$. Then, for any $1 \le i \le n$,

$$\det(A) = \sum_{k=1}^{n} a_{ik} \det(A_{ik}) \qquad (i^{th} \text{ row expansion})$$

$$\det(A) = \sum_{k=1}^{n} a_{ki} \det(A_{ki}) \qquad (i^{th} \text{ column expansion})$$

Thm: Let $A \in M_{n\times n}(\mathbb{R})$. Then A is invertible if and only if $\det(A) \neq 0$.

Ex: Note that

$$\begin{vmatrix} 1 & 3 & 4 \\ 2 & 0 & 1 \\ 0 & 0 & 2 \end{vmatrix} = (-3)\begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} = (-3)(2)(2) = -12 \neq 0.$$

Hence $\begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ is invertible. ∎

Thm: Elementary matrices have determinants

① $\det[R_i \longleftrightarrow R_j] = -1$

② $\det[\lambda R_i \longrightarrow R_i] = \lambda$

③ $\det[R_i + \lambda R_j \longrightarrow R_i] = 1.$

Ex: In the 3×3 case, we have

① $\det[R_2 \longleftrightarrow R_3] = \det\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$ ✓

② $\det[\lambda R_2 \longrightarrow R_2] = \det\begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} = \lambda$ ✓

③ $\det[R_1 + 2R_2 \longrightarrow R_1] = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1$ ✓

Thm: Let $A, B \in M_{n \times n}(\mathbb{R})$. Then

$$\det(AB) = \det(A)\det(B).$$

Thm: Let $A \in GL_n(\mathbb{R})$. Then

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

pf: Note that

$$\det(A^{-1})\det(A) = \det(A^{-1}A) = \det(I) = 1$$

so that

$$\det(A^{-1}) = \frac{1}{\det(A)}. \quad ▨$$

Def: Let $A \in M_{n\times n}(\mathbb{R})$. The cofactor matrix of A is the $n\times n$ matrix C whose entries are

$$[C]_{ij} = C_{ij}.$$

That is,

$$C = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}.$$

Def: The adjoint of $A \in M_{n\times n}(\mathbb{R})$ is the $n\times n$ matrix $\operatorname{adj}(A) = C^T$. That is,

$$\operatorname{adj}(A) = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

Ex: For

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

we have

$$C_{11} = d \qquad C_{12} = -c$$

$$C_{21} = -b \qquad C_{22} = a.$$

Hence

$$C = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}, \quad \mathrm{adj}(A) = C^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Thm: Let $A \in M_{n\times n}(\mathbb{R})$. Then

$$A \operatorname{adj}(A) = \operatorname{adj}(A) A = \det(A) I.$$

Ex: For

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \operatorname{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

We have

$$A \operatorname{adj}(A) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

$$= \det(A) I. \quad \blacksquare$$

Thm: Let $A \in GL_n(\mathbb{R})$. Then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A).$$

Ex: For

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

we have

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \quad \blacksquare$$

Thm: (Cramer's Rule) Let $A \in GL_n(\mathbb{R})$ and consider the system $AX = B$. Let A_i be the matrix obtained by replacing the i^{th} column of A by B. Then

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \quad \ldots, \quad x_n = \frac{\det(A_n)}{\det(A)}.$$

Ex: Consider $AX = B$ where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

Then

$$\det(A_1) = \det\begin{bmatrix} b_1 & b \\ b_2 & d \end{bmatrix} = b_1 d - b_2 b$$

$$\det(A_2) = \det\begin{bmatrix} a & b_1 \\ c & b_2 \end{bmatrix} = b_2 a - b_1 c.$$

Cramer's Rule then gives

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{b_1 d - b_2 b}{ad - bc}$$

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{b_2 a - b_1 c}{ad - bc}.$$

Now, check:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \dfrac{b_1 d - b_2 b}{ad - bc} \\[2ex] \dfrac{b_2 a - b_1 c}{ad - bc} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{1}{ad-bc}\left[a\left(b_1 d - b_2 b\right) + b\left(b_2 a - b_1 c\right)\right] \\[3ex] \dfrac{1}{ad-bc}\left[c\left(b_1 d - b_2 b\right) + d\left(b_2 a - b_1 c\right)\right] \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{1}{ad-bc}\, b_1 (ad - bc) \\[3ex] \dfrac{1}{ad-bc}\, b_2 (ad - bc) \end{bmatrix}$$

$$= \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}. \qquad \blacksquare$$

<u>Ex</u>: Solve the system

$$x e^{2t} \sin t - y e^{2t} \cos t = 1$$

$$2x e^{2t} \cos t + 2 y e^{2t} \sin t = t.$$

for x and y.

<u>Solution</u>: Cramer's rule gives

$$x = \frac{\det(A_1)}{\det(A)}, \quad y = \frac{\det(A_2)}{\det(A)}$$

where

$$A = \begin{bmatrix} \cancel{x}\, e^{2t} \sin t & -e^{2t} \cos t \\ \cancel{2\cos t}\; 2 e^{2t} \cos t & 2 e^{2t} \sin t \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 1 & -e^{2t} \cos t \\ t & 2 e^{2t} \sin t \end{bmatrix}$$

$$A_2 = \begin{bmatrix} e^{2t} \sin t & 1 \\ 2 e^{2t} \cos t & t \end{bmatrix}. \quad \blacksquare$$