Name: Solutions

MATH 107.01
EXAM II

This is a closed book closed notes exam. The use of calculators is not permitted. Show all work and clearly indicate your final answers. If all your work is not shown, you will not receive full credit. The last two pages are for scratch work. If you run out of room for a problem feel free to use the scratch paper, but be sure to be clear about where the work is for each problem. The point value of each question is indicated in parentheses. Make sure that you sign the Duke Honor Code at the bottom of this page. Good luck!

Problem	Points	Score
1	8	
2	12	
3	12	
4	14	
5	18	
6	10	
7	8	
8	6	
9	12	
Total	100	

Honor Code: I have completed this exam in the spirit of the Duke Community Standard and have neither given nor received aid.

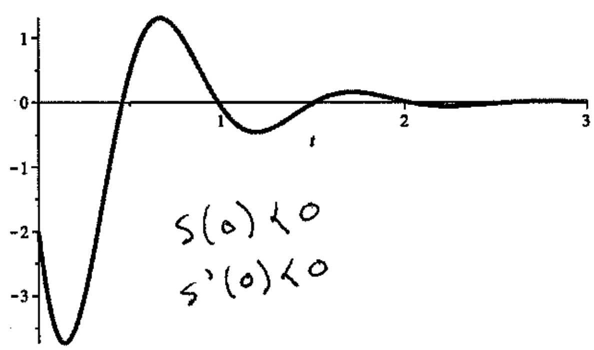
Signature: ______________________________

Problem 1 (8 Points). An object of mass 1 is hung on a spring of spring constant 40 and is suspended in oil with friction coefficient 4. No external force is applied and the object is given an initial velocity of 25 from its equilibruim position.

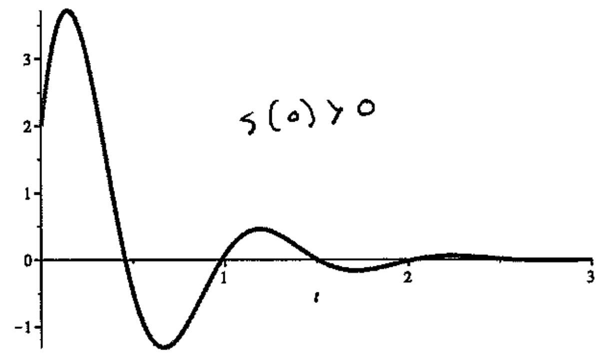
(a) Circle the differential equation satisfied by the position $s(t)$ of the object.

$40s'' + 4s' + s = 0$	$4s'' + 40s' + s = 0$
$4s'' + s' + 40s = 0$	$s'' + 4s' + 40s = 0$
$40s'' + s' + 4s = 0$	$s'' + 40s' + 4s = 0$

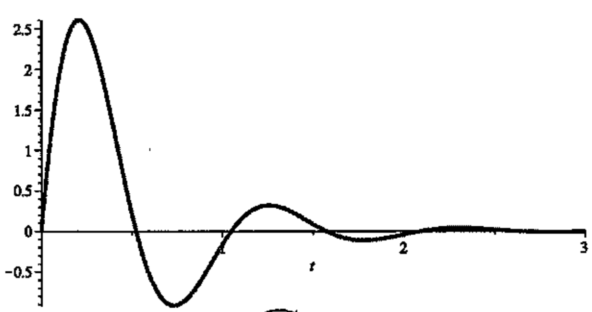
(b) Circle the graph depicting the motion of the object.

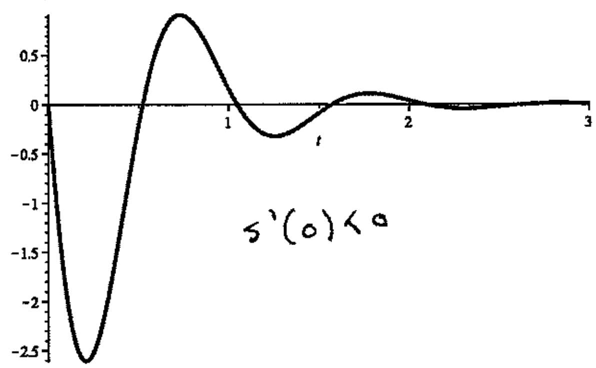


(a)



(b)


(c)



(d)

$m = 1$

$f = 4$

$k = 40$

Problem 2 (12 Points). Solve the initial value problem

$$\begin{aligned} y''' - y' &= 0 \\ y(0) &= 1 \\ y'(0) &= 1 \\ y''(0) &= 0. \end{aligned}$$

$$p(\lambda) = \lambda^3 - \lambda = \lambda(\lambda^2 - 1) = \lambda(\lambda+1)(\lambda-1)$$

$$\Rightarrow y = c_1 + c_2 e^{-x} + c_3 e^{x}$$

Imposing initial conditions gives

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow[R_3 + R_2 \to R_3]{-R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 2 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right]$$

$$\xrightarrow[R_2 + R_3 \to R_2]{R_1 - R_3 \to R_1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right] \xrightarrow{R_1 - R_2 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow c_1 = 1, \quad c_2 = -\frac{1}{2}, \quad c_3 = \frac{1}{2}$$

$$\boxed{y = 1 - \frac{1}{2}e^{-x} + \frac{1}{2}e^{x}}$$

Problem 3 (12 Points). Find the general solution to the system

$$Y'(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} Y(t).$$

$\det \begin{bmatrix} \lambda & 1 \\ -1 & \lambda \end{bmatrix} = \lambda^2+1 = (\lambda+i)(\lambda-i) \Rightarrow$ e-vals are $\lambda_1 = -i$, $\lambda_2 = i$

$E_i = \text{null} \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}$

$\left[\begin{array}{cc|c} i & 1 & 0 \\ -1 & i & 0 \end{array}\right] \xrightarrow[-R_2 \to R_2]{-iR_1 \to R_1} \left[\begin{array}{cc|c} 1 & -i & 0 \\ 1 & -i & 0 \end{array}\right] \xrightarrow{R_2 - R_1 \to R_2} \left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow E_i = \text{Span}\left\{\begin{bmatrix} i \\ 1 \end{bmatrix}\right\}$

Thm from class $\Rightarrow E_{-i} = \left\{\begin{bmatrix} -i \\ 1 \end{bmatrix}\right\}$

So, $\dim(E_i) + \dim(E_{-i}) = 2 \Rightarrow \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is diagonalizable with $P = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$, $\Lambda = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

Now, $Z = \begin{bmatrix} e^{ix} \\ 0 \end{bmatrix} = \begin{bmatrix} \cos x + i \sin x \\ 0 \end{bmatrix}$ is a sol to $Y' = \Lambda Y$

$\Rightarrow PZ = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \cos x + i \sin x \\ 0 \end{bmatrix} = \begin{bmatrix} i\cos x - \sin x \\ \cos x + i \sin x \end{bmatrix} = \begin{bmatrix} -\sin x \\ \cos x \end{bmatrix} + i \begin{bmatrix} \cos x \\ \sin x \end{bmatrix}$ solves $Y' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} Y$.

Hence $Y_1 = \begin{bmatrix} -\sin x \\ \cos x \end{bmatrix}$, $Y_2 = \begin{bmatrix} \cos x \\ \sin x \end{bmatrix}$ are sols to $Y' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} Y$.

General solution is ~~c_0~~ $Y = c_1 Y_1 + c_2 Y_2$.

Problem 4 (14 Points). Let $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ where $b \neq 0$.

(*a*) Show that the eigenvalues of A are $\lambda_1 = a + b$ and $\lambda_2 = a - b$.

$$\det \begin{bmatrix} \lambda - a & -b \\ -b & \lambda - a \end{bmatrix} = (\lambda - a)^2 + b^2 = \lambda^2 - 2a\lambda + a^2 + b^2$$

roots are $\lambda = \dfrac{2a \pm \sqrt{4a^2 - 4(a^2+b^2)}}{2} = a \pm \sqrt{a^2 - a^2 + b^2}$

$= a \pm b$ ✓

(*b*) Find bases for the eigenspaces of A.

$E_{a+b} = \text{null} \begin{bmatrix} b & -b \\ -b & b \end{bmatrix}$

$$\left[\begin{array}{cc|c} b & -b & 0 \\ -b & b & 0 \end{array}\right] \xrightarrow{R_2 + R_1 \to R_2} \left[\begin{array}{cc|c} b & -b & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{b}R_1 \to R_1} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow E_{a+b} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

$E_{a-b} = \text{null} \begin{bmatrix} -b & -b \\ -b & -b \end{bmatrix}$

$$\left[\begin{array}{cc|c} -b & -b & 0 \\ -b & -b & 0 \end{array}\right] \xrightarrow{R_2 - R_1 \to R_2} \left[\begin{array}{cc|c} -b & -b & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{-\frac{1}{b}R_1 \to R_1} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow E_{a-b} = \text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

(*c*) Is A diagonalizable? If so, find a matrix P and a diagonal matrix Λ such that $\Lambda = P^{-1}AP$.

Yes. $\dim(E_{a+b}) + \dim(E_{a-b}) = 1 + 1 = 2$ ✓

$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $\Lambda = \begin{bmatrix} a+b & 0 \\ 0 & a-b \end{bmatrix}$

Problem 5 (18 Points). Let α and β be the bases for P_1 given by

$$\alpha = \{x, 1\}$$
$$\beta = \{x+1, x-1\}$$

and let $\mathbf{1}, T : P_1 \to P_1$ be the linear transformations defined by

$$\mathbf{1}(ax+b) = ax+b$$
$$T(ax+b) = (a+2b)\,x + (2a+b)\,.$$

(*a*) Show that $\mathbf{1}T = T\mathbf{1} = T$.

$$(\mathbb{1}T)(ax+b) = \mathbb{1}\big(T(ax+b)\big) = T(ax+b) \checkmark$$

$$(T\mathbb{1})(ax+b) = T\big(\mathbb{1}(ax+b)\big) = T(ax+b) \checkmark$$

(*b*) Find $[\mathbf{1}]^{\alpha}_{\beta}$ and $[\mathbf{1}]^{\beta}_{\alpha}$.

$$\mathbb{1}(x+1) = x+1$$
$$\mathbb{1}(x-1) = x-1$$
$$\Rightarrow [\mathbb{1}]^{\beta\alpha}_{\beta} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$[\mathbb{1}]^{\beta}_{\alpha} = \left([\mathbb{1}]^{\alpha}_{\beta}\right)^{-1} = \frac{1}{-2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$$

(*c*) Find $[T]^{\alpha}_{\alpha}$.

$$T(x) = x+2$$
$$T(1) = 2x+1$$
$$\Rightarrow [T]^{\alpha}_{\alpha} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

(*d*) Express each of $[T]_\beta^\beta$, $[T]_\alpha^\beta$, and $[T]_\beta^\alpha$ in terms of $[\mathbf{1}]_\alpha^\beta$, $[\mathbf{1}]_\beta^\alpha$, and $[T]_\alpha^\alpha$. Do not substitute in the matrices you computed in (*b*) and (*c*), just do this symbolically.

$$[T]_\beta^\beta = [\mathbb{1}]_\alpha^\beta [T]_\alpha^\alpha [\mathbb{1}]_\beta^\alpha$$

$$[T]_\alpha^\beta = [\mathbb{1}]_\alpha^\beta [T]_\alpha^\alpha$$

$$[T]_\beta^\alpha = [T]_\alpha^\alpha [\mathbb{1}]_\beta^\alpha$$

(*d*) Is T diagonalizable? Why or why not?

Yes. $[T]_\alpha^\alpha = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ which is diagonalizable by Problem 4.

Problem 6 (10 Points). Let $A, B \in M_{n\times n}(\mathbb{R})$ be similar. Show that A and B have the same characteristic polynomial.

pf: Since A is similar to B, there exists a $P \in GL_n(\mathbb{R})$ such that $A = P^{-1}BP$. It follows that

$$\begin{aligned} p_A(\lambda) &= \det(\lambda I - A) = \det(\lambda P^{-1}P - P^{-1}BP) = \det(P^{-1}(\lambda I - B)P) \\ &= \det(P^{-1})\det(\lambda I - B)\det(B) = \det(P^{-1})\det(P)\,p_B(\lambda) \\ &= \det(P^{-1}P)\,p_B(\lambda) = \det(I)\,p_B(\lambda) = p_B(\lambda). \quad \blacksquare \end{aligned}$$

Problem 7 (8 Points). Write the differential equation

$$y^{(3)} + 2xy'' + e^x y = 5$$

as a system of first order differential equations in matrix form.

$v_1 = y \qquad v_1' = v_2$

$v_2 = y' \qquad v_2' = v_3$

$v_3 = y'' \qquad v_3' = -e^x v_1 - 2x v_3 + 5$

$$\begin{bmatrix} v_1' \\ v_2' \\ v_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -e^x & 0 & -2x \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$$

Problem 8 (6 Points). A linear transformation $T : V \to W$ is called **injective** if $T(v) = T(w)$ implies $v = w$.

(*a*) Suppose that T is injective. Show that $\ker(T)$ consists only of the zero vector.

Hint: Start by letting v be an arbitrary vector in $\ker(T)$ and then show that $v = \mathbf{0}$.

<u>pf</u>: Let $v \in \mathrm{Ker}(T)$. Then $T(v) = \vec{0}$ so that $T(v) = T(\vec{0})$. Since T is injective, it follows that $v = \vec{0}$. ∎

(*b*) Suppose that $\ker(T)$ consists only of the zero vector. Show that T is injective.

<u>pf</u>: Suppose $T(v) = T(w)$. Then $T(v) - T(w) = \vec{0}$. Since T is linear, it follows that $T(v-w) = \vec{0}$ so that $v - w \in \mathrm{Ker}(T)$. Since $\mathrm{Ker}(T)$ consists only of the zero vector, it follows that $v - w = \vec{0}$. Hence $v = w$ so that T is injective. ∎

Problem 9 (12 Points). Find a particular solution to the nonhomogeneous system

$$Y'(t) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} Y(t) + \begin{bmatrix} 4 \\ 6 \end{bmatrix}.$$

$$Y_p = M \int M^{-1} G \, dx$$

$$= \begin{bmatrix} e^{2x} & 0 \\ 0 & e^{3x} \end{bmatrix} \int \begin{bmatrix} e^{-2x} & 0 \\ 0 & e^{-3x} \end{bmatrix} \begin{bmatrix} 4 \\ 6 \end{bmatrix} dx$$

$$= \begin{bmatrix} e^{2x} & 0 \\ 0 & e^{3x} \end{bmatrix} \int \begin{bmatrix} 4e^{-2x} \\ 6e^{-3x} \end{bmatrix} dx$$

$$= \begin{bmatrix} e^{2x} & 0 \\ 0 & e^{3x} \end{bmatrix} \begin{bmatrix} \int 4e^{-2x} dx \\ \int 6e^{-3x} dx \end{bmatrix}$$

$$= \begin{bmatrix} e^{2x} & 0 \\ 0 & e^{3x} \end{bmatrix} \begin{bmatrix} -2e^{-2x} \\ -2e^{-3x} \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$