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MATH 107.01
EXAM I

This is a closed book closed notes exam. The use of calculators is not permitted. Show all work and clearly indicate your final answers. If all your work is not shown, you will not receive full credit. The last two pages are for scratch work. If you run out of room for a problem feel free to use the scratch paper, but be sure to be clear about where the work is for each problem. The point value of each question is indicated in parentheses. Make sure that you sign the Duke Honor Code at the bottom of this page. Good luck!

Problem	**Points**	**Score**
1	6	
2	5	
3	10	
4	10	
5	12	
6	8	
7	15	
8	12	
9	10	
10	12	
Total	100	

Honor Code: I have completed this exam in the spirit of the Duke Community Standard and have neither given nor received aid.

Signature: ______________________________

Problem 1 (6 points)**.** Circle all of the following matrices that are in reduced row echelon form.

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$E = \begin{bmatrix} 0 & 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 17 & 0 \\ 0 & 0 & 0 & 0 & 14 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad F = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Problem 2 (5 points)**.** Use Cramer's rule to find the solution to the system

$$\begin{aligned} x + 4y &= 2 \\ 2x + 7y &= 3. \end{aligned}$$

Problem 3 (10 points). Consider again the system

$$x + 4y = 2$$
$$2x + 7y = 3.$$

(a) Check your answer to Problem 2 by using elementary row operations to reduce the system to its reduced row echelon form.

(b) Use your row reduction to part (a) to write write A^{-1} as a product of elementary matrices where $A = \begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}$.

Problem 4 (10 points)**.** Let $V = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$.

(a) Use elementary row operations to show that $\det(V) = (b-a)(c-a)(c-b)$.
Hint: The identity $x^2 - y^2 = (x+y)(x-y)$ should be useful here.

(b) Find conditions on a, b, and c so that V is invertible. Explain your answer carefully.

Problem 5 (12 points)**.** Given a vector space V, let $v_1, v_2, \ldots, v_n \in V$ be linearly independent.

(a) Prove that $v_1 - v_n, v_2 - v_n, \ldots, v_{n-1} - v_n$ are linearly independent.

(b) Can $\operatorname{Span}\{v_1, v_2, \ldots, v_n\} = \operatorname{Span}\{v_1 - v_n, v_2 - v_n, \ldots, v_{n-1} - v_n\}$? Why or why not?

Problem 6 (8 points)**.** Let $A \in M_{m\times n}(\mathbb{R})$. Show that $\operatorname{rank}(A) \le \min\{m, n\}$.

Problem 7 (15 points)**.** Each of the following statements is false. Explain why (provide a counterexample where necessary).

(a) There is a 3×7 matrix A for which $\dim\big(\text{row}\,(A)\big) = \dim\big(\text{null}\,(A)\big)$.

(b) If $A + B$ is symmetric, then both A and B are symmetric.

(c) A product AB may be noninvertible even if both A and B are invertible.

(d) All invertible matrices A and B satisfy $AB = BA$.

(e) The rank of a 9×4 matrix A is 7.

Problem 8 (12 points)**.** Prove that each of the following subcollections of $M_{n\times n}(\mathbb{R})$ is a subspace of $M_{n\times n}(\mathbb{R})$.

(a) The subcollection $\mathrm{Sym}_n(\mathbb{R})$ of all symmetric $n \times n$ matrices.

(b) The subcollection $\mathrm{Skew}_n(\mathbb{R})$ of all skew-symmetric $n \times n$ matrices (an $n \times n$ matrix A is **skew-symmetric** if $A^\top = -A$).

Problem 9 (10 points)**.** By inspecting the equation $A^\top = A$, one may show that every symmetric 3×3 matrix A is of the form

$$A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}.$$

Using this formula, one may then show that the collection

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$E_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad E_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad E_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

forms a basis for the vector space $\mathrm{Sym}_3(\mathbb{R})$ of all symmetric 3×3 matrices.

(a) Use the equation $A^\top = -A$ to find a similar formula for an arbitrary skew-symmetric matrix A.

(b) Use your formula from part (a) to find a basis for the vector space $\mathrm{Skew}_3(\mathbb{R})$ of all 3×3 skew-symmetric matrices. You do **not** need to prove your answer.

Hint: Your basis should consist of three vectors.

Problem 10 (12 points)**.** Let β be the basis $\{1, x, x^2, x^3\}$ for the vector space P_3 and let $v \in P_3$ be given by $v = \lambda_0 + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3$.

(a) Find $[v]_\beta$.

(b) Compute $D \cdot [v]_\beta$ where $D = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(c) Let w be the P_3-vector satisfying $[w]_\beta = D \cdot [v]_\beta$. Find a formula for w.

(d) Let $dv = w$. What familiar polynomial is dv?

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