Name: Solutions

MATH 107.01
EXAM I

This is a closed book closed notes exam. The use of calculators is not permitted. Show all work and clearly indicate your final answers. If all your work is not shown, you will not receive full credit. The last two pages are for scratch work. If you run out of room for a problem feel free to use the scratch paper, but be sure to be clear about where the work is for each problem. The point value of each question is indicated in parentheses. Make sure that you sign the Duke Honor Code at the bottom of this page. Good luck!

Problem	Points	Score
1	6	
2	5	
3	10	
4	10	
5	12	
6	8	
7	15	
8	12	
9	10	
10	12	
Total	100	

Honor Code: I have completed this exam in the spirit of the Duke Community Standard and have neither given nor received aid.

Signature: ______________________

Problem 1 (6 points). Circle all of the following matrices that are in reduced row echelon form.

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$E = \begin{bmatrix} 0 & 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 17 & 0 \\ 0 & 0 & 0 & 0 & 14 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad F = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Problem 2 (5 points). Use Cramer's rule to find the solution to the system

$$x + 4y = 2$$
$$2x + 7y = 3.$$

Here, we have

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 2 & 4 \\ 3 & 7 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}.$$

Cramer's rule then gives

$$x = \frac{\det(A_1)}{\det(A)} = \frac{7(2) - 3(4)}{1(7) - 2(4)} = \frac{14 - 12}{7 - 8} = \frac{2}{-1} = -2$$

$$y = \frac{\det(A_2)}{\det(A)} = \frac{1(3) - 2(2)}{-1} = \frac{3 - 4}{-1} = \frac{-1}{-1} = 1. \quad \blacksquare$$

Problem 3 (10 points). Consider again the system

$$x + 4y = 2$$
$$2x + 7y = 3.$$

(*a*) Check your answer to Problem 2 by using elementary row operations to reduce the system to its reduced row echelon form.

$$\left[\begin{array}{cc|c} 1 & 4 & 2 \\ 2 & 7 & 3 \end{array}\right] \xrightarrow{R_2 - 2R_1 \to R_2} \left[\begin{array}{cc|c} 1 & 4 & 2 \\ 0 & -1 & -1 \end{array}\right] \xrightarrow{-R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 4 & 2 \\ 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 - 4R_2 \to R_1} \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 1 \end{array}\right]$$

$$\Rightarrow \operatorname{rref} \left[\begin{array}{cc|c} 1 & 4 & 2 \\ 2 & 7 & 3 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 1 \end{array}\right].$$

$$\Rightarrow \quad x = -2, \quad y = 1. \quad \blacksquare$$

(*b*) Use your row reduction to part (*a*) to write write A^{-1} as a product of elementary matrices where $A = \begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}$.

$$R_2 - 2R_1 \to R_1 \rightsquigarrow \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = E_1$$

$$-R_2 \to R_2 \rightsquigarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = E_2$$

$$R_1 - 4R_2 \to R_1 \rightsquigarrow \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} = E_3$$

$$\Rightarrow A^{-1} = E_3 E_2 E_1. \quad \blacksquare$$

Problem 4 (10 points). Let $V = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$.

(*a*) Use elementary row operations to show that $\det(V) = (b-a)(c-a)(c-b)$.
Hint: The identity $x^2 - y^2 = (x+y)(x-y)$ should be useful here.

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \xrightarrow[R_3 - R_1 \to R_3]{R_2 - R_1 \to R_2} \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$$

$$\begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & (b+a)(b-a) \\ 0 & c-a & (c+a)(c-a) \end{vmatrix} = 1 \cdot \begin{vmatrix} b-a & (b+a)(b-a) \\ c-a & (c+a)(c-a) \end{vmatrix}$$

$$= (b-a)(c+a)(c-a) - (c-a)(b+a)(b-a)$$

$$= (b-a)(c-a)\{(c+a) - (b+a)\}$$

$$= (b-a)(c-a)(c-b).$$

$$\Rightarrow \det(V) = (b-a)(c-a)(c-b). \quad \blacksquare$$

(*b*) Find conditions on *a*, *b*, and *c* so that *V* is invertible. Explain your answer carefully.

V invertible $\Leftrightarrow$ $\det(V) \neq 0$ $\Leftrightarrow$ $(b-a)(c-a)(c-b) \neq 0$ $\Leftrightarrow$ $a \neq b, a \neq c, b \neq c$.

Problem 5 (12 points). Given a vector space V, let $v_1, v_2, \ldots, v_n \in V$ be linearly independent.

(a) Prove that $v_1 - v_n, v_2 - v_n, \ldots, v_{n-1} - v_n$ are linearly independent.

pf: Consider $\lambda_1(v_1 - v_n) + \lambda_2(v_2 - v_n) + \cdots + \lambda_{n-1}(v_{n-1} - v_n) = \vec{0}$. This gives $\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_{n-1} v_{n-1} + (-\lambda_1 - \lambda_2 - \cdots - \lambda_{n-1}) v_n = \vec{0}$. Since $v_1, \ldots, v_n$ are independent, $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = -\lambda_1 - \lambda_2 - \cdots - \lambda_{n-1} = 0$. In particular, $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = 0$ so that our original equation has only the trivial solution. Hence $v_1 - v_n, v_2 - v_n, \ldots, v_{n-1} - v_n$ are linearly independent. ∎

(b) Can Span $\{v_1, v_2, \ldots, v_n\}$ = Span $\{v_1 - v_n, v_2 - v_n, \ldots, v_{n-1} - v_n\}$? Why or why not?

$\dim(\text{Span}\{v_1, \ldots, v_n\}) = n$ but $\dim(\text{Span}\{v_1 - v_n, v_2 - v_n, \ldots, v_{n-1} - v_n\}) = n-1$

So, $\text{Span}\{v_1, v_2, \ldots, v_n\} \neq \text{Span}\{v_1 - v_n, \ldots, v_{n-1} - v_n\}$.

Problem 6 (8 points). Let $A \in M_{m \times n}(\mathbb{R})$. Show that rank $(A) \leq \min\{m, n\}$.

$\text{rank}(A) = \dim(\text{row}(A)) = \dim(\text{col}(A))$.

Since $\text{row}(A)$ is spanned by m vectors, $\dim(\text{row}(A)) \leq m$

Since $\text{col}(A)$ is spanned by n vectors, $\dim(\text{col}(A)) \leq n$.

Hence $\text{rank}(A) \leq \min\{m, n\}$. ∎

Problem 7 (15 points). Each of the following statements is false. Explain why (provide a counterexample where necessary).

(a) There is a 3×7 matrix A for which $\dim(\text{row}(A)) = \dim(\text{null}(A))$.

Rank-Nullity theorem $\Rightarrow \dim(\text{row}(A)) + \dim(\text{null}(A)) = 7$.

So, one of $\dim(\text{row}(A))$ and $\dim(\text{null}(A))$ must be even and one must be odd. Hence $\dim(\text{row}(A)) \neq \dim(\text{col}(A))$.

(b) If $A + B$ is symmetric, then both A and B are symmetric.

Let A be non symmetric and $B = -A$. Then $A + B = A - A = 0$, which is symmetric, but A and B are not symmetric.

(c) A product AB may be noninvertible even if both A and B are invertible.

If A and B are invertible, then

$\det(AB) = \det(A)\det(B) \neq 0$ ~~and~~ so AB is invertible

(d) All invertible matrices A and B satisfy $AB = BA$.

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ are invert but

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ while $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$.

(e) The rank of a 9×4 matrix A is 7.

By Problem 6, $\text{rank}(A) \leq \min\{9, 4\} = 4 < 7$.

Problem 8 (12 points). Prove that each of the following subcollections of $M_{n\times n}(\mathbb{R})$ is a subspace of $M_{n\times n}(\mathbb{R})$.

(*a*) The subcollection $\mathrm{Sym}_n(\mathbb{R})$ of all symmetric $n \times n$ matrices.

pf: Note that

$$(\lambda_1 A_1 + \lambda_2 A_2)^T = (\lambda_1 A_1)^T + (\lambda_2 A_2)^T$$
$$= \lambda_1 A_1^T + \lambda_2 A_2^T$$
$$= \lambda_1 A_1 + \lambda_2 A_2$$

so that $\lambda_1 A_1 + \lambda_2 A_2 \in \mathrm{Sym}_n(\mathbb{R})$ whenever $\lambda_1, \lambda_2 \in \mathbb{R}$ and $A_1, A_2 \in \mathrm{Sym}_n(\mathbb{R})$. Hence $\mathrm{Sym}_n(\mathbb{R})$ is a subspace by the one-step subspace test. ∎

(*b*) The subcollection $\mathrm{Skew}_n(\mathbb{R})$ of all skew-symmetric $n \times n$ matrices (an $n \times n$ matrix A is **skew-symmetric** if $A^\top = -A$).

pf: Note that

$$(\lambda_1 A_1 + \lambda_2 A_2)^T = (\lambda_1 A_1)^T + (\lambda_2 A_2)^T$$
$$= \lambda_1 A_1^T + \lambda_2 A_2^T$$
$$= -\lambda_1 A_1 - \lambda_2 A_2$$
$$= -(\lambda_1 A_1 + \lambda_2 A_2)$$

so that $\lambda_1 A_1 + \lambda_2 A_2 \in \mathrm{Skew}_n(\mathbb{R})$ whenever $\lambda_1, \lambda_2 \in \mathbb{R}$ and $A_1, A_2 \in \mathrm{Skew}_n(\mathbb{R})$. By the one-step ~~subgro~~ subspace test, $\mathrm{Skew}_n(\mathbb{R})$ is a subspace. ∎

Problem 9 (10 points). By inspecting the equation $A^\top = A$, one may show that every symmetric 3×3 matrix A is of the form

$$A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}.$$

Using this formula, one may then show that the collection

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$E_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad E_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad E_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

forms a basis for the vector space $\mathrm{Sym}_3(\mathbb{R})$ of all symmetric 3×3 matrices.

(*a*) Use the equation $A^\top = -A$ to find a similar formula for an arbitrary skew-symmetric matrix A.

$$A = \begin{bmatrix} 0 & b & c \\ -b & 0 & e \\ -c & -e & 0 \end{bmatrix}$$

(*b*) Use your formula from part (*a*) to find a basis for the vector space $\mathrm{Skew}_3(\mathbb{R})$ of all 3×3 skew-symmetric matrices. You do **not** need to prove your answer.

Hint: Your basis should consist of three vectors.

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

Problem 10 (12 points). Let β be the basis $\{1, x, x^2, x^3\}$ for the vector space P_3 and let $v \in P_3$ be given by $v = \lambda_0 + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3$.

(a) Find $[v]_\beta$.

$$[v]_\beta = \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}$$

(b) Compute $D \cdot [v]_\beta$ where $D = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

$$D \cdot [v]_\beta = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ 2\lambda_2 \\ 3\lambda_3 \\ 0 \end{bmatrix}$$

(c) Let w be the P_3-vector satisfying $[w]_\beta = D \cdot [v]_\beta$. Find a formula for w.

$$w = \lambda_1 \cdot 1 + 2\lambda_2 x + 3\lambda_3 x^2$$

(d) Let $dv = w$. What familiar polynomial is dv?

dv is the derivative of v.