MATH 107.01
HOMEWORK #1 SOLUTIONS

Problem 1.1.2. *Solve the system*

$$\begin{aligned} 2x + y - 2z &= 0 \\ 2x - y - 2z &= 0 \\ x + 2y - 4z &= 0 \end{aligned}$$

Solution. Note that

$$\text{rref}\left[\begin{array}{ccc|c} 2 & 1 & -2 & 0 \\ 2 & -1 & -2 & 0 \\ 1 & 2 & -4 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Hence the only solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

□

Problem 1.1.8. *Solve the system*

$$\begin{aligned} x_1 + x_2 - x_3 + 2x_4 &= 1 \\ x_1 + x_2 - x_3 - x_4 &= -1 \\ x_1 + 2x_2 + x_3 + 2x_4 &= -1 \\ 2x_1 + 2x_2 + x_3 + x_4 &= 2 \end{aligned}$$

Solution. Note that

$$\text{rref}\left[\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 1 \\ 1 & 1 & -1 & -1 & -1 \\ 1 & 2 & 1 & 2 & -1 \\ 2 & 2 & 1 & 1 & 2 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 11/3 \\ 0 & 1 & 0 & 0 & -10/3 \\ 0 & 0 & 1 & 0 & 2/3 \\ 0 & 0 & 0 & 1 & 2/3 \end{array}\right]$$

Hence the only solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 11/3 \\ -10/3 \\ 2/3 \\ 2/3 \end{bmatrix}.$$

□

Problem 1.1.15. *Solve the system*

$$\begin{aligned} 2x_1 - x_2 - x_3 + x_4 + x_5 &= 0 \\ x_1 - x_2 + x_3 + 2x_4 - 3x_5 &= 0 \\ 3x_1 - 2x_2 - x_3 - x_4 + 2x_5 &= 0 \end{aligned}$$

Solution. Note that

$$\text{rref}\left[\begin{array}{ccccc|c} 2 & -1 & -1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 2 & -3 & 0 \\ 3 & -2 & -1 & -1 & 2 & 0 \end{array}\right] = \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 7 & -4 & 0 \\ 0 & 1 & 0 & 9 & -5 & 0 \\ 0 & 0 & 1 & 4 & -4 & 0 \end{array}\right]$$

Hence the solutions are given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 4x_5 - 7x_4 \\ 5x_5 - 9x_4 \\ 4x_5 - 4x_4 \\ x_4 \\ x_5 \end{bmatrix}.$$

□

Problem 1.1.17. *Determine conditions on a, b, and c so that the system*

$$\begin{aligned} 2x - y + 3z &= a \\ x - 3y + 2z &= b \\ x + 2y + z &= c \end{aligned}$$

has a solution.

Solution. The row reduction

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & a \\ 1 & -3 & 2 & b \\ 1 & 2 & 1 & c \end{array}\right] \xrightarrow[2R_3 \to R_3]{2R_2 \to R_2} \left[\begin{array}{ccc|c} 2 & -1 & 3 & a \\ 2 & -6 & 4 & 2b \\ 2 & 4 & 2 & 2c \end{array}\right]$$

$$\xrightarrow[R_3 - R_1 \to R_3]{R_2 - R_1 \to R_2} \left[\begin{array}{ccc|c} 2 & -1 & 3 & a \\ 0 & -5 & 1 & 2b - a \\ 0 & 5 & -1 & 2c - a \end{array}\right]$$

$$\xrightarrow{R_3 + R_2 \to R_3} \left[\begin{array}{ccc|c} 2 & -1 & 3 & a \\ 0 & -5 & 1 & 2b - a \\ 0 & 0 & 0 & -2a + 2b + 2c \end{array}\right]$$

shows that the rref of the system is consistent if and only if $-2a + 2b + 2c = 0$. Hence the system has a solution if and only if $a - b - c = 0$. □

Problem 1.1.18. *Determine conditions on a, b, and c so that the system*

$$\begin{aligned} x + 2y - z &= a \\ x + y - 2z &= b \\ 2x + y - 3z &= c \end{aligned}$$

has a solution.

Solution. The row reduction

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & a \\ 1 & 1 & -1 & b \\ 2 & 1 & -3 & c \end{array}\right] \xrightarrow[R_3 - 2R_1 \to R_3]{R_2 - R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & a \\ 0 & -1 & -2 & b - a \\ 0 & -3 & -1 & c - 2a \end{array}\right]$$

$$\xrightarrow{R_3 - 3R_2 \to R_3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & a \\ 0 & -1 & -2 & b - a \\ 0 & 0 & 5 & a - 3b + c \end{array}\right]$$

shows that the rref of the system is consistent. Hence the system has a solution for all values of a, b, and c. □

Problem 1.1.19. *Determine conditions on a, b, c, and d so that the system*

$$\begin{aligned} x_1 + x_2 + x_3 - x_4 &= a \\ x_1 - x_2 - x_3 + x_4 &= b \\ x_1 + x_2 + x_3 + x_4 &= c \\ x_1 - x_2 + x_3 + x_4 &= d \end{aligned}$$

has a solution.

Solution. The row reduction

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & a \\ 1 & -1 & -1 & 1 & b \\ 1 & 1 & 1 & 1 & c \\ 1 & -1 & 1 & 1 & d \end{array}\right] \xrightarrow[R_4 - R_1 \to R_4]{\substack{R_2 - R_1 \to R_2 \\ R_3 - R_1 \to R_3}} \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & a \\ 0 & -2 & -2 & 2 & b-a \\ 0 & 0 & 0 & 2 & c-a \\ 0 & -2 & 0 & 2 & d-a \end{array}\right]$$

$$\xrightarrow{R_4 - R_2 \to R_4} \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & a \\ 0 & -2 & -2 & 2 & b-a \\ 0 & 0 & 0 & 2 & c-a \\ 0 & 0 & 2 & 0 & d-2a-b \end{array}\right]$$

$$\xrightarrow{R_3 \leftrightarrow R_4} \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & a \\ 0 & -2 & -2 & 2 & b-a \\ 0 & 0 & 2 & 0 & d-2a-b \\ 0 & 0 & 0 & 2 & c-a \end{array}\right]$$

shows that the rref of the system is consistent. Hence the system has a solution for all values of a, b, c, and d. □

Problem 1.1.22. *Determine if the homogeneous system*

$$\begin{aligned} 99x_1 + \pi x_2 - \sqrt{5}x_3 &= 0 \\ 2x_1 + \sin(1)x_2 + 2x_4 &= 0 \\ 3.38x_1 - ex_3 + \ln(2)x_4 &= 0 \end{aligned}$$

has a nontrivial solution.

Solution. This is a homogeneous system in 3 linear equations in 4 variables. Since $3 < 4$, Theorem 1.1 in the book implies that this system has infinitely many non-trivial solutions. □

Problem 1.1.23. *Determine if the homogeneous system*

$$\begin{aligned} x - y + z &= 0 \\ 2x + y + 2z &= 0 \\ 3x - 5y + 3z &= 0 \end{aligned}$$

has a nontrivial solution.

Solution. Note that

$$\operatorname{rref}\left[\begin{array}{ccc:c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -5 & 3 & 0 \end{array}\right] = \left[\begin{array}{ccc:c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

That is, the system is consistent and has free variable x_3. Hence the system has infinitely many nontrivial solutions. □

Problem 1.1.26. *Give an example of a system of linear equations with more equations than variables that illustrates each of the following possibilities:*

(a) *Has exactly one solution.*
(b) *Has infinitely many solutions.*
(c) *Has no solution.*

Solution. (a) The system

$$\begin{aligned} x &= 0 \\ 2x &= 0 \end{aligned}$$

has more equations than variables and has $x = 0$ as a unique solution.

(b) The system

$$\begin{aligned} x + y &= 0 \\ 2x + 2y &= 0 \\ 3x + 3y &= 0 \end{aligned}$$

has more equations than variables. Furthermore,

$$\operatorname{rref}\left[\begin{array}{cc:c} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 3 & 3 & 0 \end{array}\right] = \left[\begin{array}{cc:c} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

That is, the system is consistent and has free variable y. Hence the system has infinitely many solutions.

(c) The system

$$\begin{aligned} x &= 0 \\ x &= 2 \end{aligned}$$

has more equations than variables. Moreover, it has no solutions since $0 \neq 2$. □