MATH 107.01
HOMEWORK #2 SOLUTIONS

Problem 1.2.5. *Compute $A - 4B$ where*

$$A = \begin{bmatrix} 1 & 3 \\ 3 & -1 \\ 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 \\ -3 & -2 \\ 0 & 4 \end{bmatrix}.$$

Solution. Compute

$$\begin{aligned} A - 4B &= \begin{bmatrix} 1 & 3 \\ 3 & -1 \\ 2 & -1 \end{bmatrix} - 4 \begin{bmatrix} 2 & -1 \\ -3 & -2 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & -1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} -8 & 4 \\ 12 & 8 \\ 0 & 16 \end{bmatrix} \\ &= \begin{bmatrix} 1-8 & 3+4 \\ 3+12 & -1+8 \\ 2+0 & -1+16 \end{bmatrix} = \begin{bmatrix} -7 & 7 \\ 15 & 7 \\ 2 & 15 \end{bmatrix}. \end{aligned}$$

□

Problem 1.2.9. *Compute EF where*

$$E = \begin{bmatrix} 1 & -3 & 5 \\ 2 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & -1 & 4 \\ 2 & -3 & 6 \\ 1 & 0 & 1 \end{bmatrix}.$$

Solution. Use the formulae[1]

$$\begin{aligned} [EF]_{11} &= \sum_{i=1}^{3} e_{1i} f_{i1} \\ &= (1)(1) + (-3)(2) + (5)(1) \\ &= 0 \end{aligned} \qquad \begin{aligned} [EF]_{12} &= \sum_{i=1}^{3} e_{1i} f_{i2} \\ &= (1)(-1) + (-3)(-3) + (5)(0) \\ &= 8 \end{aligned} \qquad \begin{aligned} [EF]_{13} &= \sum_{i=1}^{3} e_{1i} f_{i3} \\ &= (1)(4) + (-3)(6) + (5)(1) \\ &= -9 \end{aligned}$$

$$\begin{aligned} [EF]_{21} &= \sum_{i=1}^{3} e_{2i} f_{i1} \\ &= (2)(1) + (1)(2) + (-1)(1) \\ &= 3 \end{aligned} \qquad \begin{aligned} [EF]_{22} &= \sum_{i=1}^{3} e_{2i} f_{i2} \\ &= (2)(-1) + (1)(-3) + (-1)(0) \\ &= -5 \end{aligned} \qquad \begin{aligned} [EF]_{23} &= \sum_{i=1}^{3} e_{2i} f_{i3} \\ &= (2)(4) + (1)(6) + (-1)(1) \\ &= 13 \end{aligned}$$

$$\begin{aligned} [EF]_{31} &= \sum_{i=3}^{3} e_{1i} f_{i1} \\ &= (1)(1) + (1)(2) + (0)(1) \\ &= 3 \end{aligned} \qquad \begin{aligned} [EF]_{32} &= \sum_{i=1}^{3} e_{3i} f_{i2} \\ &= (1)(-1) + (1)(-3) + (0)(0) \\ &= -4 \end{aligned} \qquad \begin{aligned} [EF]_{33} &= \sum_{i=1}^{3} e_{3i} f_{i3} \\ &= (1)(4) + (1)(6) + (0)(1) \\ &= 10 \end{aligned}$$

to compute

$$EF = \begin{bmatrix} 1 & -3 & 5 \\ 2 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 4 \\ 2 & -3 & 6 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 8 & -9 \\ 3 & -5 & 13 \\ 3 & -4 & 10 \end{bmatrix}.$$

□

[1] Note, you were not required to use the formulae in your solutions. I include them here for the sake of demonstrating the definition of matrix multiplication.

Problem 1.2.11. *Compute AE where*

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 2 & -1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & -3 & 5 \\ 2 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Solution. Note that $A \in M_{3\times 2}(\mathbb{R})$ while $E \in M_{3\times 3}(\mathbb{R})$. Since $2 \neq 3$, the product AE cannot be formed. □

Problem 1.2.12. *Compute EA where*

$$E = \begin{bmatrix} 1 & -3 & 5 \\ 2 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 2 & -1 \end{bmatrix}.$$

Solution. Use the formulae

$$\begin{aligned} [EA]_{11} &= \sum_{i=1}^{3} e_{1i}a_{i1} \\ &= (1)(1) + (-3)(3) + (5)(2) \\ &= 2 \end{aligned} \qquad \begin{aligned} [EA]_{12} &= \sum_{i=1}^{3} e_{1i}a_{i2} \\ &= (1)(2) + (-3)(-1) + (5)(-1) \\ &= 0 \end{aligned}$$

$$\begin{aligned} [EA]_{21} &= \sum_{i=1}^{3} e_{2i}a_{i1} \\ &= (2)(1) + (1)(3) + (-1)(2) \\ &= 3 \end{aligned} \qquad \begin{aligned} [EA]_{22} &= \sum_{i=1}^{3} e_{1i}a_{i2} \\ &= (2)(2) + (1)(-1) + (-1)(-1) \\ &= 4 \end{aligned}$$

$$\begin{aligned} [EA]_{31} &= \sum_{i=1}^{3} e_{3i}a_{i1} \\ &= (1)(1) + (1)(3) + (0)(2) \\ &= 4 \end{aligned} \qquad \begin{aligned} [EA]_{32} &= \sum_{i=1}^{3} e_{1i}a_{i2} \\ &= (1)(2) + (1)(-1) + (0)(-1) \\ &= 1 \end{aligned}$$

to compute

$$EA = \begin{bmatrix} 1 & -3 & 5 \\ 2 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 3 & 4 \\ 4 & 1 \end{bmatrix}.$$ □

Problem 1.2.14. *Compute $B(C + D)$ where*

$$B = \begin{bmatrix} 2 & -1 \\ -3 & -2 \\ 0 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & -1 \\ 1 & 5 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 1 \\ 3 & -1 \end{bmatrix}.$$

Solution. Compute

$$\begin{aligned} B(C + D) &= \begin{bmatrix} 2 & -1 \\ -3 & -2 \\ 0 & 4 \end{bmatrix} \left(\begin{bmatrix} 2 & -1 \\ 1 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 3 & -1 \end{bmatrix} \right) = \begin{bmatrix} 2 & -1 \\ -3 & -2 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -4 \\ -8 & -8 \\ 4 & 16 \end{bmatrix}. \end{aligned}$$ □

Problem 1.2.18. *Compute* A^3 *where*

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 2 & -1 \end{bmatrix}.$$

Solution. Note that $A \in M_{3\times 2}(\mathbb{R})$. Since $2 \neq 3$, the product A^2 cannot be formed. Since $A^3 = A^2 A$, it follows that the product A^3 cannot be formed. □

Problem 1.2.20. *Write the system*

$$\begin{aligned} x_1 - 3x_2 + x_3 - 5x_4 &= 2 \\ x_1 + x_2 - x_3 + x_4 &= 1 \\ x_1 - x_2 - x_3 + 6x_4 &= 6 \end{aligned}$$

in the matrix form $AX = B$.

Solution. Let

$$A = \begin{bmatrix} 1 & -3 & 1 & -5 \\ 1 & 1 & -1 & 1 \\ 1 & -2 & -1 & 6 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix}.$$

Then the equation $AX = B$ yields the desired system. □

Problem 1.2.21. *Write the matrix equation*

$$\begin{bmatrix} 2 & -2 & 5 & 7 \\ 4 & 5 & -11 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 12 \\ -3 \end{bmatrix}$$

as a system of equations.

Solution. Multiplying the matrices gives

$$\begin{bmatrix} 2x_1 - 2x_2 + 5x_3 + 7x_4 \\ 4x_1 + 5x_2 - 11x_3 + 3x_4 \end{bmatrix} = \begin{bmatrix} 12 \\ -3 \end{bmatrix}.$$

This corresponds to the system

$$\begin{aligned} 2x_1 - 2x_2 + 5x_3 + 7x_4 &= 12 \\ 4x_1 + 5x_2 - 11x_3 + 3x_4 &= -3. \end{aligned}$$

□

Problem 1.2.23. *Suppose that* A *and* B *are* $n \times n$ *matrices.*

(a) *Show that* $(A + B)^2 = A^2 + AB + BA + B^2$.

(b) *Explain why* $(A + B)^2$ *is not equal to* $A^2 + 2AB + B^2$ *in general.*

Solution. (a) Use the distributive laws (Theorem 1.3) to compute

$$\begin{aligned} (A + B)^2 &= (A + B)(A + B) = (A + B)A + (A + B)B = A^2 + BA + AB + B^2 \\ &= A^2 + AB + BA + B^2 \end{aligned}$$

as desired.

(b) This is because $AB \neq BA$ in general. For example, take

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then

$$(A+B)^2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

while

$$A^2 + 2AB + B^2 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}. \qquad \square$$

Problem 1.2.25. *Given matrices A, B, and C, let $\lambda \in \mathbb{R}$. Show that*

(3) $(A+B)\,C = AC + BC$
(4) $\lambda\,(AB) = (\lambda A)\,B = A\,(\lambda B)$

whenever the indicated sums and products are defined.

Solution. (3) Here, $A, B \in M_{l\times m}(\mathbb{R})$ and $C \in M_{m\times n}(\mathbb{R})$. Then

$$\begin{aligned}\left[(A+B)\,C\right]_{ij} &= \sum_{k=1}^{m} [A+B]_{ik}\,c_{kj} = \sum_{k=1}^{m} (a_{ik}+b_{ik})\,c_{kj} = \sum_{k=1}^{m} (a_{ik}c_{kj} + b_{ik}c_{kj}) \\ &= \sum_{k=1}^{m} a_{ik}c_{kj} + \sum_{k=1}^{m} b_{ik}c_{kj} = [AC]_{ij} + [BC]_{ij} = [AC+BC]_{ij}\end{aligned}$$

so that $(A+B)\,C = AC + BC$.

(4) Here, $A \in M_{l\times m}(\mathbb{R})$ and $B \in M_{m\times n}(\mathbb{R})$.

For the first equality, note that

$$\begin{aligned}\left[\lambda\,(AB)\right]_{ij} &= \lambda\,[AB]_{ij} = \lambda \sum_{k=1}^{m} a_{ik}b_{kj} = \sum_{k=1}^{m} (\lambda a_{ik})\,b_{kj} = \sum_{k=1}^{m} [\lambda A]_{ik}\,b_{kj} \\ &= \left[(\lambda A)\,B\right]_{ij}\end{aligned}$$

so that $\lambda\,(AB) = (\lambda A)\,B$.

For the second equality, note that

$$\begin{aligned}\left[(\lambda A)\,B\right]_{ij} &= \sum_{k=1}^{m} [\lambda A]_{ik}\,b_{kj} = \sum_{k=1}^{m} (\lambda a_{ik})\,b_{kj} = \sum_{k=1}^{m} a_{ik}\,(\lambda b_{kj}) = \sum_{k=1}^{m} a_{ik}\,[\lambda B]_{kj} \\ &= \left[A\,(\lambda B)\right]_{ij}\end{aligned}$$

so that $(\lambda A)\,B = A\,(\lambda B)$. $\square$

Problem 1.2.28. *Give an example of two matrices A and B for which $AB = \mathbf{O}$ with $A \neq \mathbf{O}$ and $B \neq \mathbf{O}$.*

Solution. Let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then $A \neq \mathbf{O}$, $B \neq \mathbf{O}$, and

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{O}. \qquad \square$$

Problem 1.2.29. (*a*) *Suppose that $A \in M_{1\times n}(\mathbb{R})$ is the row vector*

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$$

and $B \in M_{n\times l}(\mathbb{R})$. *View* B *as the column of row vectors*

$$B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix}$$

where $B_1, B_2, \ldots, B_n$ *are the rows of* B. *Show that*

$$AB = a_1B_1 + a_2B_2 + \cdots + a_nB_n.$$

(b) *Use the result of part* (a) *to find* AB *form*

$$A = \begin{bmatrix} -2 & 1 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 1 & 0 \\ 2 & 1 & 1 \\ 4 & -1 & 2 \end{bmatrix}.$$

Solution. (a) Write

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$$
$$B = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1l} \\ B_{21} & B_{22} & \cdots & B_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ B_{n1} & B_{n2} & \cdots & B_{nl} \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix}$$

and note that $AB \in M_{1\times l}(\mathbb{R})$ so that $[AB]_{ij} = [AB]_{1j}$. Then compute

$$\begin{aligned} [AB]_{ij} = [AB]_{1j} &= \sum_{k=1}^{n} a_{1k}B_{1j} \\ &= a_{11}B_{1j} + a_{12}B_{2j} + \cdots + a_{1n}B_{nj} \\ &= a_1B_{1j} + a_2B_{2j} + \cdots + a_nB_{nj} \\ &= [a_1B_1 + a_2B_2 + \cdots + a_nB_n]_{ij}. \end{aligned}$$

That is, the two $1 \times l$ matrices AB and $a_1B_1 + a_2B_2 + \cdots + a_nB_n$ have the same entries. Hence

$$AB = a_1B_1 + a_2B_2 + \cdots + a_nB_n$$

as advertised.

(b) Using part (a), we have

$$\begin{aligned} AB &= -2\begin{bmatrix} -1 & 1 & 0 \end{bmatrix} + 1\begin{bmatrix} 2 & 1 & 1 \end{bmatrix} + 6\begin{bmatrix} 4 & -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -2 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 24 & -6 & 12 \end{bmatrix} \\ &= \begin{bmatrix} 2+2+24 & -2+1-6 & 0+1+12 \end{bmatrix} \\ &= \begin{bmatrix} 28 & -7 & 13 \end{bmatrix}. \end{aligned}$$

□

Problem 1.2.30. (a) *Suppose that* $B \in M_{n\times 1}(\mathbb{R})$ *is the column vector*

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

and that $A \in M_{m\times n}(\mathbb{R})$*. View* A *as the row of column vectors*

$$A = \begin{bmatrix} A_1 & A_2 & \cdots & A_n \end{bmatrix}$$

where $A_1, A_2, \ldots, A_n$ *are the columns of* A*. Show that*

$$AB = b_1A_1 + b_2A_2 + \cdots + b_nA_n.$$

(b) *Use the result of part* (a) *to find* AB *for*

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 3 & 5 \\ 1 & 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.$$

Solution. (a) Write

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix} = \begin{bmatrix} A_1 & A_2 & \cdots & A_n \end{bmatrix}$$

$$B = \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

and note that $AB \in M_{m\times 1}(\mathbb{R})$ so that $[AB]_{ij} = [AB]_{i1}$. Then compute

$$\begin{aligned} [AB]_{ij} = [AB]_{i1} &= \sum_{k=1}^{n} A_{ik}b_{k1} \\ &= A_{i1}b_{11} + A_{i2}b_{21} + \cdots + A_{in}b_{n1} \\ &= A_{i1}b_1 + A_{i2}b_2 + \cdots + A_{in}b_n \\ &= b_1A_{i1} + b_2A_{i2} + \cdots + b_nA_{in} \\ &= [b_1A_1 + b_2A_2 + \cdots + b_nA_n]_{ij}. \end{aligned}$$

That is, the two $m \times 1$ matrices AB and $b1A_1 + b_2A_2 + \cdots + b_nA_n$ have the same entries. Hence

$$AB = b_1A_1 + b_2A_2 + \cdots + b_nA_n$$

as advertised.

(b) Use part (a) to compute

$$AB = 2\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ -3 \\ -1 \end{bmatrix} + \begin{bmatrix} -3 \\ 15 \\ 6 \end{bmatrix} = \begin{bmatrix} 6-2-3 \\ 0-3+15 \\ 2-1+6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 12 \\ 7 \end{bmatrix}. \qquad \square$$

Problem 1.2.31. *The* **trace** *of a matrix* $A \in M_{n\times n}(\mathbb{R})$ *is defined as*

$$\operatorname{tr}(A) = \sum_{i=1}^{n} [A]_{ii}.$$

Find $\operatorname{tr}(A)$ *where*

$$A = \begin{bmatrix} 5 & 0 & -4 \\ 2 & -11 & 6 \\ 2 & 10 & 3 \end{bmatrix}.$$

Solution. Compute

$$\operatorname{tr}(A) = \sum_{i=1}^{3} [A]_{ii} = [A]_{11} + [A]_{22} + [A]_{33} = 5 + (-11) + 3 = -3.$$

□

Problem 1.2.32. *Prove the following where* $A, B \in M_{n\times n}(\mathbb{R})$ *and* $c \in \mathbb{R}$.

(a) $\operatorname{tr}(A+B) = \operatorname{tr}(A) + \operatorname{tr}(B)$

(b) $\operatorname{tr}(cA) = c\operatorname{tr}(A)$

(c) $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

Solution. (a) Note that

$$\operatorname{tr}(A+B) = \sum_{i=1}^{n} [A+B]_{ii} = \sum_{i=1}^{n} (a_{ii} + b_{ii}) = \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} = \operatorname{tr}(A) + \operatorname{tr}(B)$$

as advertised.

(b) Note that

$$\operatorname{tr}(cA) = \sum_{i=1}^{n} [cA]_{ii} = \sum_{i=1}^{n} c a_{ii} = c \sum_{i=1}^{n} a_{ii} = c\operatorname{tr}(A)$$

as advertised.

(c) Note that

$$\begin{aligned} \operatorname{tr}(AB) = \sum_{i=1}^{n} [AB]_{ii} &= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} b_{ji} \right) = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} a_{ij} b_{ji} \right) \\ &= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} b_{ji} a_{ij} \right) = \sum_{j=1}^{n} [BA]_{jj} = \operatorname{tr}(BA) \end{aligned}$$

as advertised. □