MATH 107.01
HOMEWORK #3 SOLUTIONS

Problem 1.3.1. *Let*

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}.$$

Find A^{-1} or determine that A is not invertible.

Solution. The row reduction

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ -3 & 1 & 0 & 1 \end{array}\right] \xrightarrow{R_2+3R_1\to R_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 7 & 3 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{7}R_2\to R_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 3/7 & 1/7 \end{array}\right]$$

$$\xrightarrow{R_1-2R_2\to R_1} \left[\begin{array}{cc|cc} 1 & 0 & 1/7 & -2/7 \\ 0 & 1 & 3/7 & 1/7 \end{array}\right]$$

gives

$$A^{-1} = \begin{bmatrix} 1/7 & -2/7 \\ 3/7 & 1/7 \end{bmatrix}.$$ □

Problem 1.3.6. *Let*

$$A = \begin{bmatrix} 0 & -1 & 3 \\ 0 & -4 & 1 \\ 2 & -1 & 3 \end{bmatrix}.$$

Find A^{-1} or determine that A is not invertible.

Solution. The row reduction

$$\left[\begin{array}{ccc|ccc} 0 & -1 & 3 & 1 & 0 & 0 \\ 0 & -4 & 1 & 0 & 1 & 0 \\ 2 & -1 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 - R_3 \to R_1} \left[\begin{array}{ccc|ccc} -2 & 0 & 0 & 1 & 0 & -1 \\ 0 & -4 & 1 & 0 & 1 & 0 \\ 2 & -1 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 + R_1 \to R_3} \left[\begin{array}{ccc|ccc} -2 & 0 & 0 & 1 & 0 & -1 \\ 0 & -4 & 1 & 0 & 1 & 0 \\ 0 & -1 & 3 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 - \frac{1}{4} R_2 \to R_3} \left[\begin{array}{ccc|ccc} -2 & 0 & 0 & 1 & 0 & -1 \\ 0 & -4 & 1 & 0 & 1 & 0 \\ 0 & 0 & 11/4 & 1 & -1/4 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 - \frac{4}{11} R_3 \to R_2} \left[\begin{array}{ccc|ccc} -2 & 0 & 0 & 1 & 0 & -1 \\ 0 & -4 & 0 & -4/11 & 12/11 & 0 \\ 0 & 0 & 11/4 & 1 & -1/4 & 0 \end{array}\right]$$

$$\xrightarrow[\frac{4}{11} R_3 \to R_3]{\substack{-\frac{1}{2} R_1 \to R_1 \\ -\frac{1}{4} R_2 \to R_2}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & 0 & 1/2 \\ 0 & 1 & 0 & 1/11 & -3/11 & 0 \\ 0 & 0 & 1 & 4/11 & -1/11 & 0 \end{array}\right]$$

gives

$$A^{-1} = \begin{bmatrix} -1/2 & 0 & 1/2 \\ 1/11 & -3/11 & 0 \\ 4/11 & -1/11 & 0 \end{bmatrix}. \qquad \square$$

Problem 1.3.7. *Let*

$$A = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 1 & 2 & -1 & -1 \\ 1 & -4 & 1 & 5 \\ 3 & 1 & 1 & 6 \end{bmatrix}.$$

Find A^{-1} or determine that A is not invertible.

Solution. The row reduction

$$\left[\begin{array}{rrrr|rrrr} 1 & -1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & -1 & -1 & 0 & 1 & 0 & 0 \\ 1 & -4 & 1 & 5 & 0 & 0 & 1 & 0 \\ 3 & 1 & 1 & 6 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{R_2 - R_1 \to R_2 \\ R_3 - R_1 \to R_3 \\ R_4 - 3R_1 \to R_4}} \left[\begin{array}{rrrr|rrrr} 1 & -1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 3 & -2 & -3 & -1 & 1 & 0 & 0 \\ 0 & -3 & 0 & 3 & -1 & 0 & 1 & 0 \\ 0 & 4 & -2 & 0 & -3 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\substack{R_3 + R_2 \to R_3 \\ R_4 - \frac{4}{3}R_2 \to R_4}} \left[\begin{array}{rrrr|rrrr} 1 & -1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 3 & -2 & -3 & -1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & -2 & 1 & 1 & 0 \\ 0 & 0 & 2/3 & 4 & -5/3 & -4/3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_4 + \frac{1}{3}R_3 \to R_4} \left[\begin{array}{rrrr|rrrr} 1 & -1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 3 & -2 & -3 & -1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & -2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 4 & -7/3 & -1 & 1/3 & 1 \end{array}\right]$$

$$\xrightarrow{\substack{R_1 - \frac{1}{2}R_4 \to R_1 \\ R_2 + \frac{3}{4}R_4 \to R_2}} \left[\begin{array}{rrrr|rrrr} 1 & -1 & 1 & 0 & 13/6 & 1/2 & -1/6 & -1/2 \\ 0 & 3 & -2 & 0 & -11/4 & 1/4 & 1/4 & 3/4 \\ 0 & 0 & -2 & 0 & -2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 4 & -7/3 & -1 & 1/3 & 1 \end{array}\right]$$

$$\xrightarrow{\substack{R_1 + \frac{1}{2}R_3 \to R_1 \\ R_2 - R_3 \to R_2}} \left[\begin{array}{rrrr|rrrr} 1 & -1 & 0 & 0 & 7/6 & 1 & 1/3 & -1/2 \\ 0 & 3 & 0 & 0 & -3/4 & -3/4 & -3/4 & 3/4 \\ 0 & 0 & -2 & 0 & -2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 4 & -7/3 & -1 & 1/3 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 + \frac{1}{3}R_2 \to R_1} \left[\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & 11/12 & 3/4 & 1/12 & -1/4 \\ 0 & 3 & 0 & 0 & -3/4 & -3/4 & -3/4 & 3/4 \\ 0 & 0 & -2 & 0 & -2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 4 & -7/3 & -1 & 1/3 & 1 \end{array}\right]$$

$$\xrightarrow{\substack{\frac{1}{3}R_2 \to R_2 \\ -\frac{1}{2}R_3 \to R_3 \\ \frac{1}{4}R_4 \to R_4}} \left[\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & 11/12 & 3/4 & 1/12 & -1/4 \\ 0 & 1 & 0 & 0 & -1/4 & -1/4 & -1/4 & 1/4 \\ 0 & 0 & 1 & 0 & 1 & -1/2 & -1/2 & 0 \\ 0 & 0 & 0 & 1 & -7/12 & -1/4 & 1/12 & 1/4 \end{array}\right]$$

gives

$$A^{-1} = \begin{bmatrix} 11/12 & 3/4 & 1/12 & -1/4 \\ -1/4 & -1/4 & -1/4 & 1/4 \\ 1 & -1/2 & -1/2 & 0 \\ -7/12 & -1/4 & 1/12 & 1/4 \end{bmatrix}. \qquad \square$$

Problem 1.3.10. *Use the inverse of the matrix in Exercise 7 to solve the system*

$$\begin{aligned} x_1 - x_2 + x_3 + 2x_4 &= 3 \\ x_1 + 2x_2 - x_3 - x_4 &= 5 \\ x_1 - 4x_2 + x_3 + 5x_4 &= 1 \\ x_1 + x_2 + x_3 + 6x_4 &= 2. \end{aligned}$$

Solution. The matrix form of the system is

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ 1 & 2 & -1 & -1 \\ 1 & -4 & 1 & 5 \\ 3 & 1 & 1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 1 \\ 2 \end{bmatrix}.$$

Multiplying this equation through by

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ 1 & 2 & -1 & -1 \\ 1 & -4 & 1 & 5 \\ 3 & 1 & 1 & 6 \end{bmatrix}^{-1} = \begin{bmatrix} 11/12 & 3/4 & 1/12 & -1/4 \\ -1/4 & -1/4 & -1/4 & 1/4 \\ 1 & -1/2 & -1/2 & 0 \\ -7/12 & -1/4 & 1/12 & 1/4 \end{bmatrix}$$

gives

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 11/12 & 3/4 & 1/12 & -1/4 \\ -1/4 & -1/4 & -1/4 & 1/4 \\ 1 & -1/2 & -1/2 & 0 \\ -7/12 & -1/4 & 1/12 & 1/4 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 73/12 \\ -7/4 \\ 0 \\ -29/12 \end{bmatrix}.$$ □

Problem 1.3.16. *Show that a square matrix containing a zero row or a zero column is not invertible.*

Solution. Let $A \in M_{n\times n}(\mathbb{R})$ have a zero row. Then there exists a k such that $[A]_{ki} = 0$ for all i. It follows that

$$[AB]_{kk} = \sum_{i=1}^{n} [A]_{ki} [B]_{ik} = \sum_{i=1}^{n} 0 \, [B]_{ik} = 0$$

whenever $B \in M_{n\times n}(\mathbb{R})$. That is, $AB \neq I$ whenever $B \in M_{n\times n}(\mathbb{R})$. Hence A is not invertible.

The argument when A has a zero column is similar. □

Problem 1.3.18. *Suppose that A is a noninvertible square matrix. Show that the homogeneous system $AX = 0$ has nontrivial solutions.*

Solution. Since A is noninvertible, $\operatorname{rref}[A : 0]$ has a row of zeros. It follows that the homogeneous system $AX = 0$ has a free variable. Hence $AX = 0$ has infinitely nontrivial solutions. □

Problem 1.3.21. *Suppose that $A, B \in M_{n\times n}(\mathbb{R})$ such that AB is invertible. Show that A and B are invertible.*

Solution. Since AB is invertible, there exists a $C \in M_{n\times n}(\mathbb{R})$ such that $C(AB) = (AB)C = I$. It follows that $(CA)B = A(BC) = I$. Theorem 1.9 then implies that A and B are invertible with $A^{-1} = BC$ and $B^{-1} = CA$. □

Extra Problem. *Let*

$$A = \begin{bmatrix} 0 & -1 & 3 \\ 0 & -4 & 1 \\ 2 & -1 & 3 \end{bmatrix}.$$

Write A^{-1} as a product of elementary matrices.

Solution. From Problem 1.3.6, the row reduction

$$\left[\begin{array}{ccc|ccc} 0 & -1 & 3 & 1 & 0 & 0 \\ 0 & -4 & 1 & 0 & 1 & 0 \\ 2 & -1 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 - R_3 \to R_1} \left[\begin{array}{ccc|ccc} -2 & 0 & 0 & 1 & 0 & -1 \\ 0 & -4 & 1 & 0 & 1 & 0 \\ 2 & -1 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 + R_1 \to R_3} \left[\begin{array}{ccc|ccc} -2 & 0 & 0 & 1 & 0 & -1 \\ 0 & -4 & 1 & 0 & 1 & 0 \\ 0 & -1 & 3 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 - \frac{1}{4} R_2 \to R_3} \left[\begin{array}{ccc|ccc} -2 & 0 & 0 & 1 & 0 & -1 \\ 0 & -4 & 1 & 0 & 1 & 0 \\ 0 & 0 & 11/4 & 1 & -1/4 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 - \frac{4}{11} R_3 \to R_2} \left[\begin{array}{ccc|ccc} -2 & 0 & 0 & 1 & 0 & -1 \\ 0 & -4 & 0 & -4/11 & 12/11 & 0 \\ 0 & 0 & 11/4 & 1 & -1/4 & 0 \end{array}\right]$$

$$\xrightarrow[\frac{4}{11} R_3 \to R_3]{\substack{-\frac{1}{2} R_1 \to R_1 \\ -\frac{1}{4} R_2 \to R_2}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & 0 & 1/2 \\ 0 & 1 & 0 & 1/11 & -3/11 & 0 \\ 0 & 0 & 1 & 4/11 & -1/11 & 0 \end{array}\right]$$

gives

$$A^{-1} = \begin{bmatrix} -1/2 & 0 & 1/2 \\ 1/11 & -3/11 & 0 \\ 4/11 & -1/11 & 0 \end{bmatrix}.$$

These row reductions correspond to elementary matrices

$$R_1 - R_3 \to R_1 \rightsquigarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_1$$

$$R_3 + R_1 \to R_3 \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = E_2$$

$$R_3 - \tfrac{1}{4}R_2 \to R_3 \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/4 & 1 \end{bmatrix} = E_3$$

$$R_2 - \tfrac{4}{11}R_3 \to R_2 \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4/11 \\ 0 & 0 & 1 \end{bmatrix} = E_4$$

$$-\tfrac{1}{2}R_1 \to R_1 \rightsquigarrow \begin{bmatrix} -1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_5$$

$$-\tfrac{1}{4}R_2 \to R_2 \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/4 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_6$$

$$\tfrac{4}{11}R_3 \to R_3 \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4/11 \end{bmatrix} = E_7.$$

This gives

$$A^{-1} = E_7E_6E_5E_4E_3E_2E_1.$$

□