MATH 107.01
HOMEWORK #4 SOLUTIONS

Problem 1.4.4. *Let*

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Find $\left(A^{-1}\right)^4$.

Solution. Use Theorem 1.11 to compute

$$\left(A^{-1}\right)^4 = \left(\operatorname{diag}\left(-1,-2,1\right)^{-1}\right)^4 = \operatorname{diag}\left(-1,-\frac{1}{2},1\right)^4 = \operatorname{diag}\left(1,\frac{1}{16},1\right). \qquad \square$$

Problem 1.4.12. *Let*

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 1 \\ 3 & 5 \\ -4 & 1 \end{bmatrix}.$$

Find $B^\top A^\top$*, if possible.*

Solution. Use Theorem 1.13 to compute

$$B^\top A^\top = (AB)^\top = \left(\begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 3 & 5 \\ -4 & 1 \end{bmatrix}\right)^\top = \begin{bmatrix} 16 & 8 \\ -12 & -8 \end{bmatrix}^\top = \begin{bmatrix} 16 & -12 \\ 8 & -8 \end{bmatrix}.$$

$\square$

Problem 1.4.17. *Let*

$$A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 0 & 4 \\ 3 & 4 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 & -1 \\ 1 & 4 & 2 \\ -1 & 2 & 1 \end{bmatrix}.$$

Determine if $A+B$ *is symmetric.*

Solution. Note that

$$A + B = \begin{bmatrix} 4 & -2 & 2 \\ -1 & 4 & 6 \\ 2 & 6 & 6 \end{bmatrix}$$

so that

$$(A+B)^\top \begin{bmatrix} 4 & -1 & 2 \\ -2 & 4 & 6 \\ 2 & 6 & 6 \end{bmatrix}.$$

It follows that $(A+B) \neq (A+B)^\top$. Hence $A+B$ is not symmetric. $\square$

Problem 1.4.20. *Let*

$$B = \begin{bmatrix} 3 & 0 & -1 \\ 1 & 4 & 2 \\ -1 & 2 & 1 \end{bmatrix}.$$

Determine if $B^\top B$ is symmetric.

Solution. Theorem 1.13 implies

$$\left(B^\top B\right)^\top = B^\top \left(B^\top\right)^\top = B^\top B.$$

Hence $B^\top B$ is symmetric. (Note that this works for all $n \times n$ matrices). □

Problem 1.4.21. (b) *Let*

$$A = \operatorname{diag}(a_1, a_2, \ldots, a_n)$$
$$B = \operatorname{diag}(b_1, b_2, \ldots, b_n).$$

Show that $AB = \operatorname{diag}(a_1b_1, a_2b_2, \ldots, a_nb_n)$.

Solution. Since $a_{ij} = b_{ij} = 0$ for $i \neq j$, we have

$$[AB]_{ii} = \sum_{k=1}^{n} a_{ik}b_{ki} = a_{ii}bii = [\operatorname{diag}(a_1b_1, a_2b_2, \ldots, a_nb_n)]_{ii}$$

and

$$\begin{aligned} [AB]_{ij} &= \sum_{k=1}^{n} a_{ik}b_{kj} \\ &= a_{ii}b_{ij} + a_{ij}b_{jj} \\ &= a_i \cdot 0 + 0 \cdot b_j \\ &= 0 \\ &= [\operatorname{diag}(a_1b_1, a_2b_2, \ldots, a_nb_n)]_{ij} \end{aligned}$$

when $i \neq j$. Hence $AB = \operatorname{diag}(a_1b_1, a_2b_2, \ldots, a_nb_n)$. □

Problem 1.4.24. (d) *Let $A \in M_{n\times n}(\mathbb{R})$ be invertible and symmetric. Prove that A^{-1} is symmetric.*

Solution. Use Theorem 1.13 to compute $\left(A^{-1}\right)^\top = \left(A^\top\right)^{-1} = A^{-1}$. □

Problem 1.4.32. *Let*

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Find A^3.

Solution. Compute

$$\begin{aligned} A^3 &= \left(\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

□