MATH 107.01
HOMEWORK #5 SOLUTIONS

Problem 1.5.5. *Let*

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 1 & -2 \\ -3 & 2 & 1 \end{bmatrix}$$

Find $\det(A)$ *by expanding about column* 2.

Solution. Use Theorem 1.16 in the book to compute

$$\begin{aligned} \det(A) &= (-1)\begin{vmatrix} 4 & -2 \\ -3 & 1 \end{vmatrix} - (1)\begin{vmatrix} 2 & 3 \\ -3 & 1 \end{vmatrix} + (2)\begin{vmatrix} 2 & 3 \\ 4 & -2 \end{vmatrix} \\ &= (-1)\Big((4)(1) - (-3)(-2)\Big) - (1)\Big((2)(1) - (-3)(3)\Big) \\ &\qquad + (2)\Big((2)(-2) - (4)(3)\Big) \\ &= -41. \end{aligned}$$

□

Problem 1.5.8. *Find*

$$\begin{vmatrix} 2 & -1 & 5 & 6 \\ 0 & 3 & 4 & 0 \\ 0 & 1 & 5 & 2 \\ 0 & 1 & -3 & 0 \end{vmatrix}.$$

Solution. Expand about column 1 and then about column 3

$$\begin{aligned} \begin{vmatrix} 2 & -1 & 5 & 6 \\ 0 & 3 & 4 & 0 \\ 0 & 1 & 5 & 2 \\ 0 & 1 & -3 & 0 \end{vmatrix} &= (2)\begin{vmatrix} 3 & 4 & 0 \\ 1 & 5 & 2 \\ 1 & -3 & 0 \end{vmatrix} = (2)(-2)\begin{vmatrix} 3 & 4 \\ 1 & -3 \end{vmatrix} \\ &= (2)(-2)\Big((3)(-3) - (1)(4)\Big) \\ &= 52. \end{aligned}$$

□

Problem 1.5.12. *Use row operations to find*

$$\begin{vmatrix} 2 & -1 & 3 & 1 \\ -1 & 2 & -1 & 4 \\ 1 & -1 & 3 & 1 \\ 3 & 2 & -1 & 5 \end{vmatrix}.$$

Solution. By Theorem 1.20 in the book, the row reduction

$$\begin{vmatrix} 2 & -1 & 3 & 1 \\ -1 & 2 & -1 & 4 \\ 1 & -1 & 3 & 1 \\ 3 & 2 & -1 & 5 \end{vmatrix} \xrightarrow{R_1 \leftrightarrow R_3} (-1) \begin{vmatrix} 1 & -1 & 3 & 1 \\ -1 & 2 & -1 & 4 \\ 2 & -1 & 3 & 1 \\ 3 & 2 & -1 & 5 \end{vmatrix}$$

$$\xrightarrow[R_4 - 3R_1 \to R_4]{\substack{R_2 + R_1 \to R_2 \\ R_3 - 2R_1 \to R_3}} (-1) \begin{vmatrix} 1 & -1 & 3 & 3 & 1 \\ 0 & 1 & 2 & 2 & 5 \\ 0 & 1 & -3 & -3 & -1 \\ 0 & 5 & -10 & -10 & 2 \end{vmatrix}$$

$$\xrightarrow{\substack{R_3 - R_2 \to R_3 \\ R_4 - 5R_2 \to R_4}} (-1) \begin{vmatrix} 1 & -1 & 3 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & -5 & -6 \\ 0 & 0 & -20 & -23 \end{vmatrix}$$

$$\xrightarrow{R_4 - 4R_3 \to R_4} (-1) \begin{vmatrix} 1 & -1 & 3 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & -5 & -6 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

gives

$$\begin{vmatrix} 2 & -1 & 3 & 1 \\ -1 & 2 & -1 & 4 \\ 1 & -1 & 3 & 1 \\ 3 & 2 & -1 & 5 \end{vmatrix} = (-1) \begin{vmatrix} 1 & -1 & 3 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & -5 & -6 \\ 0 & 0 & 0 & 1 \end{vmatrix} = (-1)(1)(1)(-5)(1) = 5. \qquad \square$$

Problem 1.5.16. *Let $A \in M_{n\times n}(\mathbb{R})$ have rows and columns*

$$A = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{bmatrix} = \begin{bmatrix} C_1 & C_2 & \cdots & C_n \end{bmatrix}.$$

(a) *Suppose that $R_i = \lambda R_j$ where $\lambda \in \mathbb{R}$ and $i \neq j$. Show that* $\det(A) = 0$.
(a) *Suppose that $C_i = \lambda C_j$ where $\lambda \in \mathbb{R}$ and $i \neq j$. Compute* $\det(A)$.

Solution. (a) The matrix produced by applying the elementary row operation $R_i - \lambda R_j \to R_i$ to A has a row of zeros. It then follows from Corollary 1.17 and Theorem 1.20 that $\det(A) = 0$.

(b) Note that $A^\top$ has rows $R_i^\top = C_i$. So, the matrix produced by applying the elementary row operation $R_i^\top - \lambda R_j^\top \to R_i^\top$ to $A^\top$ has a row of zeros. It then follows from part (a) and Theorem 1.19 that $\det(A) = \det\left(A^\top\right) = 0$. $\square$