MATH 107.01
HOMEWORK #6 SOLUTIONS

Problem 1.6.4. *Use Theorem 1.21 to determine if the matrix*

$$A = \begin{bmatrix} 2 & -1 & -3 \\ 1 & 1 & 3 \\ 6 & 0 & 0 \end{bmatrix}$$

is invertible.

Solution. Compute

$$\det(A) = \begin{vmatrix} 2 & -1 & -3 \\ 1 & 1 & 3 \\ 6 & 0 & 0 \end{vmatrix} = 6 \begin{vmatrix} -1 & -3 \\ 1 & 3 \end{vmatrix} = 6(-3+3) = 0.$$

Hence A is not invertible. □

Problem 1.6.6. *Use the adjoint method to find the inverse of*

$$A = \begin{bmatrix} -2 & 3 \\ 1 & -2 \end{bmatrix}.$$

Solution. Corollary 1.27 gives

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \frac{1}{4-3} \begin{bmatrix} -2 & -3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ -1 & -2 \end{bmatrix}.$$ □

Problem 1.6.10. *Use Cramer's rule to solve the system*

$$\begin{aligned} 5x - 4y + z &= 2 \\ 2x - 3y - 2z &= 4 \\ 3x + y + 3z &= 2. \end{aligned}$$

Solution. This is the system $AX = B$ where

$$A = \begin{bmatrix} 5 & -4 & 1 \\ 2 & -3 & -2 \\ 3 & 1 & 3 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}.$$

Cramer's rule gives

$$x = \frac{1}{\det(A)} \begin{vmatrix} 2 & -4 & 1 \\ 4 & -3 & -2 \\ 2 & 1 & 3 \end{vmatrix} = \frac{60}{24} = \frac{5}{2}$$

$$y = \frac{1}{\det(A)} \begin{vmatrix} 5 & 2 & 1 \\ 2 & 4 & -2 \\ 3 & 2 & 3 \end{vmatrix} = \frac{48}{24} = 2$$

$$z = \frac{1}{\det(A)} \begin{vmatrix} 5 & -4 & 2 \\ 2 & -3 & 4 \\ 3 & 1 & 2 \end{vmatrix} = -\frac{60}{24} = -\frac{5}{2}.$$ □

Problem 1.6.11. *Use Cramer's rule to solve the system*

$$xe^t \sin 2t + ye^t \cos 2t = t$$
$$2xe^t \cos 2t - 2ye^t \sin 2t = t^2$$

for x and y.

Solution. This is the system

$$A = \begin{bmatrix} e^t \sin 2t & e^t \cos 2t \\ 2e^t \cos 2t & -2e^t \sin 2t \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \begin{bmatrix} t \\ t^2 \end{bmatrix}.$$

Cramer's rule then gives

$$x = \frac{1}{\det(A)} \begin{vmatrix} t & e^t \cos 2t \\ t^2 & -2e^t \sin 2t \end{vmatrix} = \frac{-te^t(2\sin 2t + t\cos 2t)}{-2e^{2t}}$$
$$y = \frac{1}{\det A} \begin{vmatrix} e^t \sin 2t & t \\ 2e^t \cos 2t & t^2 \end{vmatrix} = \frac{te^t(t\sin 2t - 2\cos 2t)}{-2e^{2t}}.$$ □

Problem 1.6.15. *Let*

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}.$$

(*a*) *Find* $\det(A)$ *and* $\det(B)$.
(*b*) *Find* $\det(AB)$, $\det\left(A^{-1}\right)$, *and* $\det\left(B^\top A^{-1}\right)$ *without finding* AB, A^{-1}, *or* $B^\top A^{-1}$.
(*c*) *Show that* $\det(A+B) \neq \det(A) + \det(B)$.

Proof. (*a*) Compute

$$\det(A) = \begin{vmatrix} 3 & -2 \\ 1 & 4 \end{vmatrix} = 12 + 2 = 14$$
$$\det(B) = \begin{vmatrix} 1 & 2 \\ -2 & 3 \end{vmatrix} = 3 + 4 = 7.$$

(*b*) Use part (*a*), Theorem 1.19, and Corollary 1.25 to compute

$$\det(AB) = \det(A)\det(B) = (14)(7) = 98$$
$$\det\left(A^{-1}\right) = \frac{1}{\det(A)} = \frac{1}{14}$$
$$\det\left(B^\top A^{-1}\right) = \det\left(B^\top\right)\det\left(A^{-1}\right) = \frac{\det(B)}{\det(A)} = \frac{7}{14} = \frac{1}{2}.$$

(*c*) Note that

$$\det(A+B) = \begin{vmatrix} 4 & 0 \\ -1 & 7 \end{vmatrix} = 28 \neq 21 = 14 + 7 = \det(A) + \det(B).$$ □

Problem 1.6.16. *Let $A, B \in M_{n\times n}(\mathbb{R})$. Show that* $\det(AB) = \det(BA)$.

Proof. Theorem 1.24 gives

$$\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA).$$ □

Problem 2.1.5. *Let V be the collection of pairs of positive real numbers of the form (x, y) with addition and scalar multiplication defined as*

$$(x_1, y_1) + (x_2, y_2) = (x_1 x_2, y_1 y_2)$$
$$\lambda (x, y) = \left(x^\lambda, y^\lambda\right).$$

Show that V is a vector space.

Solution. Verify the axioms.

Axiom 1. This holds by the computation

$$(x_1, y_1) + (x_1, y_2) = (x_1 x_2, y_1 y_2) = (x_2 x_1, y_2 y_1) = (x_2, y_2) + (x_1, y_1). \quad \checkmark$$

Axiom 2. This holds by the computation

$$\begin{aligned}(x_1, y_1) + \big((x_2, y_2) + (x_3, y_3)\big) &= (x_1, y_1) + (x_2 x_3, y_2 y_3) \\ &= \big(x_1 (x_2 x_3), y_1 (y_2 y_3)\big) \\ &= \big((x_1 x_2) x_3, (y_1 y_2) y_3\big) \\ &= (x_1 x_2, y_1 y_2) + (x_3, y_3) \\ &= \big((x_1, y_1) + (x_2, y_2)\big) + (x_3, y_3). \quad \checkmark\end{aligned}$$

Axiom 3. Here, $\mathbf{0} = (1, 1)$. To see this, compute

$$(x, y) + \mathbf{0} = (x, y) + (1, 1) = (x \cdot 1, y \cdot 1) = (x, y). \quad \checkmark$$

Axiom 4. Here, $-(x, y) = \left(x^{-1}, y^{-1}\right)$. To see this, compute

$$(x, y) + \big(-(x, y)\big) = (x, y) + \left(x^{-1}, y^{-1}\right) = \left(x \cdot x^{-1}, y \cdot y^{-1}\right) = (1, 1) = \mathbf{0}. \quad \checkmark$$

Axiom 5. This holds by the computation

$$\begin{aligned}\lambda\big((x_1, y_1) + (x_2, y_2)\big) &= \lambda (x_1 x_2, y_1 y_2) = \Big((x_1 x_2)^\lambda, (y_1 y_2)^\lambda\Big) \\ &= \left(x_1^\lambda x_2^\lambda, y_1^\lambda y_2^\lambda\right) = \left(x_1^\lambda, y_1^\lambda\right) + \left(x_2^\lambda, y_2^\lambda\right) \\ &= \lambda (x_1, y_1) + \lambda (x_2, y_2). \quad \checkmark\end{aligned}$$

Axiom 6. This holds by the computation

$$\begin{aligned}(\lambda_1 + \lambda_2)(x, y) &= \left(x^{\lambda_1 + \lambda_2}, y^{\lambda_1 + \lambda_2}\right) = \left(x^{\lambda_1} x^{\lambda_2}, y^{\lambda_1} y^{\lambda_2}\right) \\ &= \left(x^{\lambda_1}, y^{\lambda_1}\right) + \left(x^{\lambda_2}, y^{\lambda_2}\right) = \lambda_1 (x, y) + \lambda_2 (x, y). \quad \checkmark\end{aligned}$$

Axiom 7. This holds by the computation

$$\begin{aligned}\lambda_1\big(\lambda_2 (x, y)\big) &= \lambda_1 \left(x^{\lambda_2}, y^{\lambda_2}\right) = \Big(\left(x^{\lambda_2}\right)^{\lambda_1}, \left(y^{\lambda_2}\right)^{\lambda_1}\Big) \\ &= \left(x^{\lambda_1 \lambda_2}, y^{\lambda_1 \lambda_2}\right) = (\lambda_1 \lambda_2)(x, y). \quad \checkmark\end{aligned}$$

Axiom 8. This holds by the computation

$$1 (x, y) = \left(x^1, y^1\right) = (x, y). \quad \checkmark$$

Since all the axioms hold, V is a vector space. $\square$

Problem 2.1.7. *Let C be the collection of all convergent sequences of real numbers $\{a_n\}$ with addition and scalar multiplication defined by*

$$\begin{aligned}\{a_n\} + \{b_n\} &= \{a_n + b_n\} \\ \lambda\{a_n\} &= \{\lambda a_n\}.\end{aligned}$$

Determine if C is a vector space.

Solution. We will show that C is a vector space.

First, note that addition and scalar multiplication is well-defined. Indeed, if $a_n \to A$ and $b_n \to B$, then

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = A + B$$

so that addition is well-defined and

$$\lim_{n\to\infty} \lambda a_n = \lambda \lim_{n\to\infty} a_n = \lambda A$$

so that scalar multiplication is well-defined.

Now, we may verify the axioms.

Axiom 1. This holds by the computation

$$\{a_n\} + \{b_n\} = \{a_n + b_n\} = \{b_n + a_n\} = \{b_n\} + \{a_n\}. \qquad \checkmark$$

Axiom 2. This holds by the computation

$$\begin{aligned}\{a_n\} + \big(\{b_n\} + \{c_n\}\big) &= \{a_n\} + \{b_n + c_n\} = \big\{a_n + (b_n + c_n)\big\} \\ &= \big\{(a_n + b_n) + c_n\big\} = \{a_n + b_n\} + \{c_n\} \\ &= \big(\{a_n\} + \{b_n\}\big) + \{c_n\}. \qquad \checkmark\end{aligned}$$

Axiom 3. Here, $\mathbf{0} = \{\mathbf{0}_n\}$ where $\mathbf{0}_n = 0$. To see this, compute

$$\{a_n\} + \mathbf{0} = \{a_n\} + \{\mathbf{0}_n\} = \{a_n + \mathbf{0}_n\} = \{a_n + 0\} = \{a_n\}. \qquad \checkmark$$

Axiom 4. Here $-\{a_n\} = \{-a_n\}$. To see this, compute

$$\{a_n\} + \big(-\{a_n\}\big) = \{a_n\} + \{-a_n\} = \{a_n - a_n\} = \{0\} = \{\mathbf{0}_n\} = \mathbf{0}. \qquad \checkmark$$

Axiom 5. This holds by the computation

$$\begin{aligned}\lambda\big(\{a_n\} + \{b_n\}\big) &= \lambda\{a_n + b_n\} = \big\{\lambda(a_n + b_n)\big\} \\ &= \{\lambda a_n + \lambda b_n\} = \{\lambda a_n\} + \{\lambda b_n\} \\ &= \lambda\{a_n\} + \lambda\{b_n\}. \qquad \checkmark\end{aligned}$$

Axiom 6. This holds by the computation

$$\begin{aligned}(\lambda_1 + \lambda_2)\{a_n\} &= \big\{(\lambda_1 + \lambda_2)a_n\big\} = \{\lambda_1 a_n + \lambda_2 a_n\} \\ &= \{\lambda_1 a_n\} + \{\lambda_2 a_n\} = \lambda_1\{a_n\} + \lambda_2\{a_n\}. \qquad \checkmark\end{aligned}$$

Axiom 7. This holds by the computation

$$\begin{aligned}\lambda_1\big(\lambda_2\{a_n\}\big) &= \lambda_1\{\lambda_2 a_n\} = \big\{\lambda_1(\lambda_2 a_n)\big\} \\ &= \big\{(\lambda_1\lambda_2)a_n\big\} = (\lambda_1\lambda_2)\{a_n\}. \qquad \checkmark\end{aligned}$$

Axiom 8. This holds by the computation

$$1\{a_n\} = \{1 \cdot a_n\} = \{a_n\}. \qquad \checkmark$$

Since all the axioms hold, C is a vector space. □

Problem 2.1.8. *Let S denote the collection of all convergent series of real numbers $\sum_{n=1}^{\infty} a_n$ with addition and scalar multiplication defined by*

$$\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (a_n + b_n)$$
$$\lambda \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \lambda a_n.$$

Determine if S is a vector space.

Solution. We will show that S is a vector space.

First, note that addition and scalar multiplication is well-defined. Indeed, if $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$, then

$$\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n = A + B$$

so that addition is well-defined and

$$\sum_{n=1}^{\infty} \lambda a_n = \lambda \sum_{n=1}^{\infty} a_n = \lambda A$$

so that scalar multiplication is well-defined.

Now, we may verify the axioms.

Axiom 1. This holds by the computation

$$\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} (b_n + a_n) = \sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} a_n. \qquad \checkmark$$

Axiom 2. This holds by the computation

$$\begin{aligned}\sum_{n=1}^{\infty} a_n + \left(\sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} c_n\right) &= \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} (b_n + c_n) \\ &= \sum_{n=1}^{\infty} \big(a_n + (b_n + c_n)\big) \\ &= \sum_{n=1}^{\infty} (a_n + b_n) + \sum_{n=1}^{\infty} c_n \\ &= \left(\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n\right) + \sum_{n=1}^{\infty} c_n. \qquad \checkmark\end{aligned}$$

Axiom 3. Here, $\mathbf{0} = \sum_{n=1}^{\infty} \mathbf{0}_n$ where $\mathbf{0}_n = 0$. To see this, compute

$$\sum_{n=1}^{\infty} a_n + \mathbf{0} = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} \mathbf{0}_n = \sum_{n=1}^{\infty} (a_n + \mathbf{0}_n) = \sum_{n=1}^{\infty} (a_n + 0) = \sum_{n=1}^{\infty} a_n. \qquad \checkmark$$

Axiom 4. Here, $-\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-a_n)$. To see this, compute

$$\sum_{n=1}^{\infty} a_n + \left(-\sum_{n=1}^{\infty} a_n\right) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} (-a_n) = \sum_{n=1}^{\infty} (a_n - a_n) = \sum_{n=1}^{\infty} 0 = \sum_{n=1}^{\infty} \mathbf{0}_n = \mathbf{0}.$$

$\checkmark$

Axiom 5. This holds by the computation

$$\begin{aligned}\lambda\left(\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n\right) &= \lambda \sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} \lambda (a_n + b_n) \\ &= \sum_{n=1}^{\infty} (\lambda a_n + \lambda b_n) = \sum_{n=1}^{\infty} \lambda a_n + \sum_{n=1}^{\infty} \lambda b_n \\ &= \lambda \sum_{n=1}^{\infty} a_n + \lambda \sum_{n=1}^{\infty} b_n. \qquad \checkmark\end{aligned}$$

Axiom 6. This holds by the computation

$$\begin{aligned}(\lambda_1 + \lambda_2) \sum_{n=1}^{\infty} a_n &= \sum_{n=1}^{\infty} (\lambda_1 + \lambda_2) a_n = \sum_{n=1}^{\infty} (\lambda_1 a_n + \lambda_2 a_n) \\ &= \sum_{n=1}^{\infty} \lambda_1 a_n + \sum_{n=1}^{\infty} \lambda_2 a_n = \lambda_1 \sum_{n=1}^{\infty} a_n + \lambda_2 \sum_{n=1}^{\infty} a_n. \qquad \checkmark\end{aligned}$$

Axiom 7. This holds by the computation

$$\begin{aligned}\lambda_1\left(\lambda_2 \sum_{n=1}^{\infty} a_n\right) &= \lambda_1 \sum_{n=1}^{\infty} \lambda_2 a_n = \sum_{n=1}^{\infty} \lambda_1 (\lambda_2 a_n) \\ &= \sum_{n=1}^{\infty} (\lambda_1 \lambda_2) a_n = (\lambda_1 \lambda_2) \sum_{n=1}^{\infty} a_n. \qquad \checkmark\end{aligned}$$

Axiom 8. This holds by the computation

$$1 \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} 1 \cdot a_n = \sum_{n=1}^{\infty} a_n. \qquad \checkmark$$

Since all the axioms hold, S is a vector space. □

Problem 2.1.9. *Let V be the collection consisting of a single element z with addition and scalar multiplication defined by*

$$\begin{aligned} z + z &= z \\ \lambda z &= z. \end{aligned}$$

Show that V is a vector space.

Solution. Verify the axioms.

Axiom 1. This follows from the computation $z + z = z = z + z$. ✓

Axiom 2. This follows from the computation

$$z + (z + z) = z + z = z = (z + z) + z. \qquad \checkmark$$

Axiom 3. Here, $\mathbf{0} = z$. To see this, note that $z + \mathbf{0} = z + z = z$. ✓

Axiom 4. Here, $-z = z$. To see this, note that $z + (-z) = z + z = z = \mathbf{0}$. ✓

Axiom 5. This follows from the computation $\lambda (z + z) = \lambda z = z = \lambda z + \lambda z$. ✓

Axiom 6. This follows from the computation $(\lambda_1 + \lambda_2) z = z = \lambda_1 z + \lambda_2 z$. ✓

Axiom 7. This follows from the computation $\lambda_1 (\lambda_2 z) = \lambda_1 z = z = (\lambda_1 \lambda_2) z$. ✓

Axiom 8. This follows from the computation $1z = z$. ✓

Since all the axioms hold, V is a vector space. □