MATH 107.01
HOMEWORK #7 SOLUTIONS

Problem 2.2.1. *Determine which of the following sets of vectors are subspaces of* $\mathbb{R}^2$.

(c) *All vectors of the form* $\begin{bmatrix} x \\ 2-5x \end{bmatrix}$.

(d) *All vectors of the form* $\begin{bmatrix} x \\ y \end{bmatrix}$ *where* $x + y = 0$.

Solution. (c) This is not a subspace since the vector $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is not of the form $\begin{bmatrix} x \\ 2-5x \end{bmatrix}$.

(d) Note that this is the collection of all vectors of the form $\begin{bmatrix} x \\ -x \end{bmatrix}$. Furthermore, the computation

$$\lambda_1 \begin{bmatrix} x_1 \\ -x_1 \end{bmatrix} + \lambda_2 \begin{bmatrix} x_2 \\ -x_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 \\ -\lambda_1 x_1 \end{bmatrix} + \begin{bmatrix} \lambda_2 x_2 \\ -\lambda_2 x_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 + \lambda_2 x_2 \\ -\lambda_1 x_1 - \lambda_2 x_2 \end{bmatrix}$$
$$= \begin{bmatrix} \lambda_1 x_1 + \lambda_2 x_2 \\ -\left(\lambda_1 x_1 + \lambda_2 x_2\right) \end{bmatrix}$$

shows that the one-step subgroup test holds for this collection so that the collection is a subspace. □

Problem 2.2.2. *Determine which of the following sets of vectors are subspaces of* $\mathbb{R}^3$.

(b) *All vectors of the form* $\begin{bmatrix} y+z+1 \\ y \\ z \end{bmatrix}$.

(d) *All vectors of the form* $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ *where* $z = x^2 + y^2$.

Proof. (b) This is not a subspace since the vector $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is not of the form $\begin{bmatrix} y+z+1 \\ y \\ z \end{bmatrix}$.

(d) This is not a subspace since the vectors $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ are of the form $\begin{bmatrix} x \\ y \\ x^2+y^2 \end{bmatrix}$ while their sum $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$ is not of the form $\begin{bmatrix} x \\ y \\ x^2+y^2 \end{bmatrix}$. □

Problem 2.2.3. *(c) Let W be the subcollection of $\mathfrak{F}[a,b]$ consisting of all functions f in $\mathscr{C}^0[a,b]$ for which $\int_a^b f(x)\,dx = 0$. Determine if W is a subspace of $\mathfrak{F}[a,b]$.*

Solution. Note that

$$\int_a^b (\lambda_1 f_1(x) + \lambda_2 f_2(x))\;dx = \lambda_1 \int_a^b f_1(x)\,dx + \lambda_2 \int_a^b f_2(x)\,dx = \lambda_1 \cdot 0 + \lambda_2 \cdot 0 = 0$$

whenever $\lambda_1, \lambda_2 \in \mathbb{R}$ and $f_1, f_2 \in W$. That is, $\lambda_1 f_1 + \lambda_2 f_2 \in W$ whenever $\lambda_1, \lambda_2 \in \mathbb{R}$ and $f_1, f_2 \in W$. By the one-step subspace test, W is a subspace of $\mathfrak{F}[a,b]$. ☐

Problem 2.2.5. *Given $A \in M_{m\times n}(\mathbb{R})$ and nonzero $B \in \mathbb{R}^m$, let W be the subcollection of $\mathbb{R}^n$ consisting all solutions X to the system $AX = B$. Does W form a subspace of $\mathbb{R}^n$?*

Solution (Solution)**.** *Since $A\mathbf{0} = \mathbf{0} \neq B$, we have that $\mathbf{0} \notin W$. Hence W is not a subspace of $\mathbb{R}^n$.*

Problem 2.2.11. *Is* $\begin{bmatrix} 1 \\ -5 \\ -3 \end{bmatrix}$ *in* Span $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$*?*

Solution. This is equivalent to asking if the system

$$\left[\begin{array}{ccc:c} 1 & 1 & 2 & 1 \\ -1 & 1 & 0 & -5 \\ 0 & 1 & 1 & -3 \end{array}\right]$$

has a solution. Since

$$\operatorname{rref}\left[\begin{array}{ccc:c} 1 & 1 & 2 & 1 \\ -1 & 1 & 0 & -5 \\ 0 & 1 & 1 & -3 \end{array}\right] = \left[\begin{array}{ccc:c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right],$$

we see that the system is inconsistent and therefore has no solution. Hence $\begin{bmatrix} 1 \\ -5 \\ -3 \end{bmatrix}$ is not in Span $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$. ☐

Problem 2.2.12. *Is* $\begin{bmatrix} 3 & 5 \\ 3 & 4 \end{bmatrix}$ *in* Span $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$*?*

Solution. This is equivalent to asking if the system

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 1 & 1 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 1 & 0 & -1 & 4 \end{array}\right]$$

has a solution. Since

$$\operatorname{rref}\left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 1 & 1 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 1 & 0 & -1 & 4 \end{array}\right] = \left[\begin{array}{ccc:c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

we see that the system is consistent and therefore has a solution. Hence $\begin{bmatrix} 3 & 5 \\ 3 & 4 \end{bmatrix}$ is in Span $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$. ☐

Problem 2.2.13. *Is* $3x^2$ *in* Span $\left\{ x^2 - x, x^2 + x + 1, x^2 - 1 \right\}$*?*

Solution. This is equivalent to asking if the system

$$\left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 3 \end{array}\right]$$

has a solution. Since

$$\operatorname{rref} \left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 3 \end{array}\right] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

we see that this system is consistent and therefore has a solution. Hence $3x^2$ is in Span $\left\{ x^2 - x, x^2 + x + 1, x^2 - 1 \right\}$. ☐