MATH 107.01
HOMEWORK #8 SOLUTIONS

Problem 2.3.7. *Determine if the $M_{2\times 2}(\mathbb{R})$ vectors*

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

are independent.

Solution. The linear combination

$$\lambda_1 \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

gives the system

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right].$$

Since

$$\operatorname{rref} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

we see that the system only has the trivial solution $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Hence

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

are independent. □

Problem 2.3.10. *Determine if the P_3 vectors $x^3 - 1, x^2 - 1, x - 1, 1$ are linearly independent.*

Solution. The linear combination

$$\lambda_1 \left(x^3 - 1\right) + \lambda_2 \left(x^2 - 1\right) + \lambda_3 (x - 1) + \lambda_4 = \mathbf{0}$$

gives the system

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & 1 & 0 \end{array}\right].$$

Since

$$\operatorname{rref} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & 1 & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right],$$

we see that the system has only the trivial solution $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$. Hence $x^3 - 1, x^2 - 1, x - 1, 1$ are linearly independent. □

Problem 2.3.14. *Show that*

$$\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ -1 \end{bmatrix}$$

form a basis for $\mathbb{R}^3$.

Solution. Note that for every $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in \mathbb{R}^3$, the equation

$$\lambda_1 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + \lambda_3 \begin{bmatrix} 1 \\ -4 \\ -1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

gives the system

$$\begin{bmatrix} 2 & 1 & 1 \\ -1 & 3 & -4 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \tag{1}$$

Now, since

$$\begin{aligned} \det \begin{bmatrix} 2 & 1 & 1 \\ -1 & 3 & -4 \\ 0 & -1 & -1 \end{bmatrix} &= (2) \begin{vmatrix} 3 & -4 \\ -1 & -1 \end{vmatrix} - (-1) \begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} \\ &= (2)(-3-4) + (-1+1) = -14 \neq 0, \end{aligned}$$

the matrix

$$\begin{bmatrix} 2 & 1 & 1 \\ -1 & 3 & -4 \\ 0 & -1 & -1 \end{bmatrix}$$

is invertible. It follows that for every $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in \mathbb{R}^3$ the system in (1) has a unique solution. That is,

$$\operatorname{Span} \left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ -1 \end{bmatrix} \right\} = \mathbb{R}^3.$$

Moreover, the equation

$$\lambda_1 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + \lambda_3 \begin{bmatrix} 1 \\ -4 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

only has the trivial solution so that the vectors are linearly independent. Hence

$$\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ -1 \end{bmatrix}$$

form a basis for $\mathbb{R}^3$. □

Problem 2.3.17. *Show that* $x^2 + x + 1, x^2 - x + 1, x^2 - 1$ *form a basis for* P_2.

Solution. Note that for every $a_2x^2 + a_1x + a_0 \in P_2$, the equation

$$\lambda_1\left(x^2+x+1\right)+\lambda_2\left(x^2-x+1\right)+\lambda_3\left(x^2-1\right)=a_2x^2+a_1x+a_0$$

gives the system

$$\begin{bmatrix}1 & 1 & 1\\ 1 & -1 & 0\\ 1 & 1 & -1\end{bmatrix}\begin{bmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{bmatrix}=\begin{bmatrix}a_2\\ a_1\\ a_0\end{bmatrix}. \tag{2}$$

Now, since

$$\begin{aligned}\det\begin{bmatrix}1 & 1 & 1\\ 1 & -1 & 0\\ 1 & 1 & -1\end{bmatrix} &= (1)\begin{vmatrix}1 & -1\\ 1 & 1\end{vmatrix}+(-1)\begin{vmatrix}1 & 1\\ 1 & -1\end{vmatrix}\\ &= (1+1)-(-1-1)=4\neq 0,\end{aligned}$$

the matrix

$$\begin{bmatrix}1 & 1 & 1\\ 1 & -1 & 0\\ 1 & 1 & -1\end{bmatrix}$$

is invertible. That is, for every $a_2^2 + a_1^2 + a_0 \in P_2$, the system in (2) has a unique solution. It follows that Span $\left\{x^2+x+1, x^2-x+1, x^2-1\right\} = P_2$. Moreover, the equation

$$\lambda_1\left(x^2+x+1\right)+\lambda_2\left(x^2-x+1\right)+\lambda_3\left(x^2-1\right)=\mathbf{0}$$

is only solved by the trivial solution $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Hence x^2+x+1, x^2-x+1, x^2-1 form a basis for P_2. □

Problem 2.3.22. *Show that*

$$\begin{bmatrix}-1\\ 2\\ 1\end{bmatrix},\begin{bmatrix}1\\ 1\\ 0\end{bmatrix},\begin{bmatrix}0\\ 1\\ 1\end{bmatrix},\begin{bmatrix}1\\ 1\\ -1\end{bmatrix}$$

do not form a basis for $\mathbb{R}^3$.

Solution. Note that the equation

$$\lambda_1\begin{bmatrix}-1\\ 2\\ 1\end{bmatrix}+\lambda_2,\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}+\lambda_3\begin{bmatrix}0\\ 1\\ 1\end{bmatrix}+\lambda_4\begin{bmatrix}1\\ 1\\ -1\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

gives the system

$$\left[\begin{array}{cccc|c}-1 & 1 & 0 & 1 & 0\\ 2 & 1 & 1 & 1 & 0\\ 1 & 0 & 1 & -1 & 0\end{array}\right]$$

which is a homogeneous system with more variables than equations. It follows that the system has infinitely many nontrivial solutions and the vectors

$$\begin{bmatrix}-1\\ 2\\ 1\end{bmatrix},\begin{bmatrix}1\\ 1\\ 0\end{bmatrix},\begin{bmatrix}0\\ 1\\ 1\end{bmatrix},\begin{bmatrix}1\\ 1\\ -1\end{bmatrix}$$

are not linearly independent. Hence

$$\begin{bmatrix}-1\\ 2\\ 1\end{bmatrix},\begin{bmatrix}1\\ 1\\ 0\end{bmatrix},\begin{bmatrix}0\\ 1\\ 1\end{bmatrix},\begin{bmatrix}1\\ 1\\ -1\end{bmatrix}$$

do not form a basis for $\mathbb{R}^3$. $\square$

Problem 2.3.24. *Let β be the basis*

$$\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ -1 \end{bmatrix}$$

for $\mathbb{R}^3$.

(*a*) *Find $[v]_\beta$ where $v = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.*

(*b*) *Find v if $[v]_\beta = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.*

Solution. (*a*) We wish to find $\lambda_1, \lambda_2, \lambda_3$ so that

$$\lambda_1 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + \lambda_3 \begin{bmatrix} 1 \\ -4 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

Since

$$\text{rref} \left[\begin{array}{rrr|r} 2 & 1 & 1 & -1 \\ -1 & 3 & -4 & 1 \\ 0 & -1 & -1 & 0 \end{array}\right] = \left[\begin{array}{rrr|r} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 1/14 \\ 0 & 0 & 1 & -1/14 \end{array}\right],$$

we have $\lambda_1 = -1/2$, $\lambda_2 = 1/14$, and $\lambda_3 = -1/14$. That is,

$$[v]_\beta = \begin{bmatrix} -1/2 \\ 1/14 \\ -1/14 \end{bmatrix}.$$

(*b*) Here,

$$v = (-1) \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + (1) \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + (0) \begin{bmatrix} 1 \\ -4 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ -1 \end{bmatrix}.$$

$\square$

Problem 2.3.25. *Let β be the basis $x^2 + x + 1, x^2 - x + 1, x^2 - 1$ for P_2.*

(*a*) *Find $[v]_\beta$ if $v = 2x^2 + 3x$.*

(*b*) *Find v if $[v]_\beta = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.*

Solution. (*a*) We wish to find $\lambda_1, \lambda_2, \lambda_3$ such that

$$\lambda_1 \left(x^2 + x + 1\right) + \lambda_2 \left(x^2 - x + 1\right) + \lambda_3 \left(x^2 - 1\right) = 2x^2 + 3x.$$

This gives the system

$$\begin{aligned} \lambda_1 + \lambda_2 + \lambda_3 &= 2 \\ \lambda_1 - \lambda_2 &= 3 \\ \lambda_1 + \lambda_2 - \lambda_3 &= 0. \end{aligned}$$

Since

$$\operatorname{rref}\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & 1 & -1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right],$$

we have $\lambda_1 = 2$, $\lambda_2 = -1$, and $\lambda_3 = 1$. Hence

$$[v]_\beta = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}.$$

(b) Here, we have

$$v = (1)\left(x^2 + x + 1\right) + (2)\left(x^2 - x + 1\right) + (3)\left(x^2 - 1\right).$$ □

Problem 2.3.27. *Given a vector space V, let $v_1, \ldots, v_n \in V$. Show that $\mathbf{0}, v_1, \ldots, v_n$ are linearly dependent.*

Solution. Note that

$$1 \cdot \mathbf{0} + 0 \cdot v_1 + 0 \cdot v_2 + \cdots + 0 \cdot v_n = \mathbf{0}.$$

This gives a nontrivial linear combination of $\mathbf{0}, v_1, \ldots, v_n$ that gives $\mathbf{0}$. Hence $\mathbf{0}, v_1, \ldots, v_n$ are linearly dependent. □

Problem 2.3.28. *Given a vector space V, let $v_1, \ldots, v_n \in V$ be linearly independent. Then $v_1, \ldots, v_m$ are linearly independent for $m < n$.*

Solution. Consider the linear combination

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m = \mathbf{0}.$$

Then

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m + 0 \cdot v_{m+1} + \cdots + 0 \cdot v_n = \mathbf{0}.$$

Since $v_1, \ldots, v_n$ are linearly independent, it follows that $\lambda_1 = \lambda_2 = \cdots = \lambda_m = 0$. Hence $v_1, \ldots, v_m$ are linearly independent. □