MATH 107.01
HOMEWORK #9 SOLUTIONS

Problem 2.4.4. (*c*) *Determine if*

$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

form a basis for $M_{2\times 2}(\mathbb{R})$.

Solution. Considering

$$\lambda_1 \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \lambda_3 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \lambda_4 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

gives the system

$$\begin{aligned} \lambda_2 + \lambda_3 &= 0 \\ \lambda_1 + \lambda_4 &= 0 \\ \lambda_4 &= 0 \\ \lambda_1 + \lambda_2 &= 0. \end{aligned}$$

Since

$$\operatorname{rref} \left[\begin{array}{cccc:c} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{cccc:c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right],$$

we have $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$. So, these vectors are linearly independent. Furthermore, since $\dim\big(M_{2\times 2}(\mathbb{R})\big) = 4$, these vectors form a basis. □

Problem 2.4.7. *Let*

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}.$$

(*a*) *Find a basis for* null (A).
(*b*) *Find a basis for* row (A).
(*c*) *Find a basis for* col (A).
(*d*) *Determine* rank (A).

Solution. (*a*) Since

$$\operatorname{rref} \left[\begin{array}{ccc:c} 1 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 2 & 0 \end{array}\right] = \left[\begin{array}{ccc:c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

we see that null (A) is the subspace of $\mathbb{R}^3$ consisting of all vectors of the form

$$\begin{bmatrix} x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

That is,

$$\text{null}(A) = \text{Span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}\right\}.$$

Hence

$$\begin{bmatrix}1\\1\\0\end{bmatrix}$$

is a basis for $\text{null}(A)$.

(b) Since

$$\text{rref}(A) = \begin{bmatrix}1 & -1 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix},$$

we see that $\text{row}(A) = \text{row}\big(\text{rref}(A)\big)$ is spanned by

$$\begin{bmatrix}1 & -1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 1\end{bmatrix}.$$

Furthermore, these vectors are linearly independent. Hence

$$\begin{bmatrix}1 & -1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 1\end{bmatrix}$$

form a basis for $\text{row}(A)$.

(c) Since

$$\text{rref}\left(A^\top\right) = \begin{bmatrix}1 & 0 & 2\\0 & 1 & 1\\0 & 0 & 0\end{bmatrix},$$

we see that $\text{row}\left(A^\top\right)$ is spanned by

$$\begin{bmatrix}1 & 0 & 2\end{bmatrix}, \begin{bmatrix}0 & 1 & 1\end{bmatrix}$$

so that $\text{col}(A)$ is spanned by

$$\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}.$$

Furthermore, these vectors are linearly independent. Hence

$$\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}.$$

form a basis for $\text{col}(A)$.

(d) By (b), $\text{rank}(A) = \dim\big(\text{row}(A)\big) = 2$. □

Problem 2.4.18. *Given $m > n$, let $A \in M_{m\times n}(\mathbb{R})$. Show that the rows of A are linearly dependent.*

Solution. Since $\text{col}(A)$ is spanned by the n columns of A, we have

$$\dim\big(\text{row}(A)\big) = \dim\big(\text{col}(A)\big) \leq n < m.$$

So, $\text{row}(A)$ has a basis consisting of $\dim\big(\text{row}(A)\big) < m$ vectors. This means that the m rows of A are linearly dependent. □