MATH 107.01
HOMEWORK #10 SOLUTIONS

Problem 2.5.5. *Show that* $x^2-1, x^2+1, x+1$ *are linearly independent on* $\mathbb{R}$.

Solution. Note that

$$w\big(x^2-1, x^2+1, x+1\big)_{x=0} = \begin{vmatrix} x^2-1 & x^2+1 & x+1 \\ 2x & 2x & 1 \\ 2 & 2 & 0 \end{vmatrix}_{x=0} = \begin{vmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 2 & 2 & 0 \end{vmatrix} = 4 \neq 0.$$

Theorem 2.15 in the book then implies that $x^2-1, x^2+1, x+1$ are linearly independent on $\mathbb{R}$. □

Problem 2.5.6. *Show that* e^x, e^{2x}, e^{3x} *are linearly independent on* $\mathbb{R}$.

Solution. Note that

$$w\big(e^x, e^{2x}, e^{3x}\big)_{x=0} = \begin{vmatrix} e^x & e^x & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix}_{x=0} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = 2 \neq 0.$$

Theorem 2.15 in the book then implies that e^x, e^{2x}, e^{3x} are linearly independent on $\mathbb{R}$. □

Problem 2.5.7. *Show that* $e^{4x}, xe^{4x}, x^2e^{4x}$ *are linearly independent on* $\mathbb{R}$.

Solution. Note that

$$\begin{aligned} w\big(e^{4x}, xe^{4x}, x^2e^{4x}\big)_{x=0} &= \begin{vmatrix} e^{4x} & xe^{4x} & x^2e^{4x} \\ 4e^{4x} & e^{4x}+4xe^{4x} & 2xe^{4x}+4x^2e^{4x} \\ 16e^{4x} & 8e^{4x}+16xe^{4x} & 2e^{4x}+16xe^{4x}+16x^2e^{4x} \end{vmatrix}_{x=0} \\ &= \begin{vmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 8 & 2 \end{vmatrix} \\ &= 2 \\ &\neq 0. \end{aligned}$$

Theorem 2.15 in the book then implies that $e^{4x}, xe^{4x}, x^2e^{4x}$ are linearly independent on $\mathbb{R}$. □

Problem 2.5.8. *Show that* $e^x, e^x\cos x, e^x\sin x$ *are linearly independent on* $\mathbb{R}$.

Solution. Note that

$$\begin{aligned} w(e^x, e^x\cos x, e^x\sin x)_{x=0} &= \begin{vmatrix} e^x & e^x\cos x & e^x\sin x \\ e^x & e^x(\cos x - \sin x) & e^x(\sin x + \cos x) \\ e^x & -2e^x\sin x & 2e^x\cos x \end{vmatrix}_{x=0} \\ &= \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{vmatrix} \end{aligned}$$

$$= 1$$
$$\neq 0.$$

Theorem 2.15 in the book then implies that $e^x, e^x \cos x, e^x \sin x$ are linearly independent on $\mathbb{R}$. □

Problem 2.5.12. *Show that* $\sin^2 x, \cos^2 x, \cos 2x$ *are linearly dependent on* $\mathbb{R}$.

Solution. Using the trigonometric identities

$$\sin^2 x + \cos^2 x = 1$$
$$\cos 2x = 1 - 2\sin^2 x,$$

we see that

$$\begin{aligned}\sin^2 x - \cos^2 x + \cos 2x &= \left(2\sin^2 x - \sin^2 x\right) - \cos^2 x + 1 - 2\sin^2 x \\ &= -\left(\sin^2 x + \cos^2 x\right) + 1 \\ &= -1 + 1 \\ &= 0.\end{aligned}$$

Hence $\sin^2 x, \cos^2 x, \cos 2x$ are linearly dependent on $\mathbb{R}$. □

Note. We cannot use the Wronskian here because there are linearly independent collections with nonzero Wronskians (see page 108 example 2).

Problem 3.1.1. *Determine if* $y = e^x$ *solves the differential equation*

$$y'' - 2y' + y = 0. \tag{1}$$

If $y = e^x$ *is a solution to* (1), *then determine if* $y = e^x$ *satisfies the initial condition* $y(0) = -1$.

Solution. Note that for $y = e^x$, $y = y' = y'' = e^x$. So, for $y = e^x$, we have

$$y'' - 2y' + y = e^x - 2e^x + e^x = 0.$$

Hence $y = e^x$ solves (1). Obviously, $e^0 = 1 \neq -1$, so $y = e^x$ does not satisfy the initial condition $y(0) = -1$. □

Problem 3.1.18. *Consider the differential equation*

$$y' = xy - xy^3 = xy\,(1-y)\,(1+y)\,. \tag{2}$$

(a) *Determine the equilibrium solutions to* (2).
(b) *On each region determined by the equilibrium solutions, determine when the graph is increasing, decreasing, concave up, and concave down.*

Solution. (a) The equilibrium solutions to (2) are the constants y such that

$$xy\,(1-y)\,(1+y) = 0$$

for all x. Dividing through by x gives

$$y\,(1-y)\,(1+y) = 0.$$

Hence the equilibrium solutions are $y = 0$ and $y = \pm 1$.
(b) See Figure 3.1. □

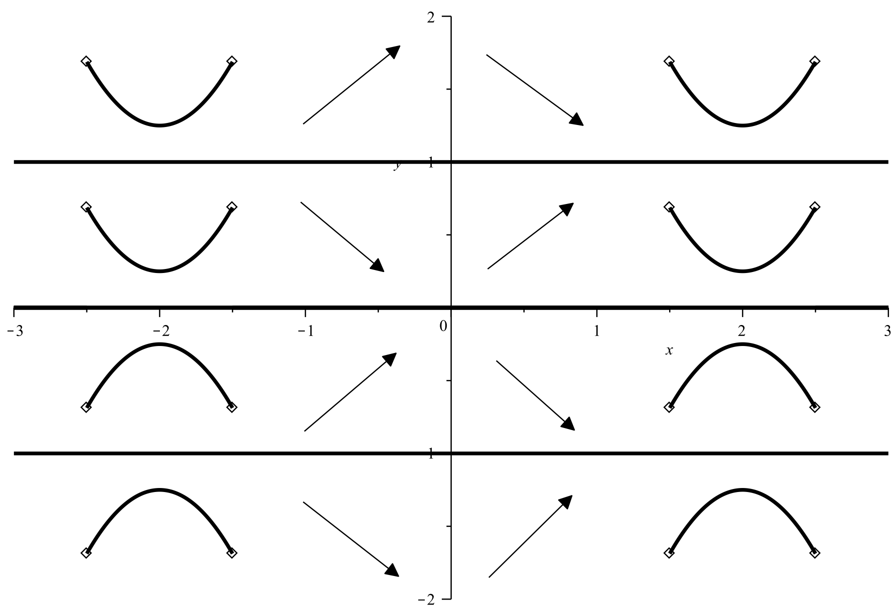

FIGURE 3.1. The derivative behavior and concavity to (2).