MATH 107.01
HOMEWORK #11 SOLUTIONS

Problem 4.1.2. *Determine if the differential equation*

$$e^y y'' - 4y' + 5xy = 0 \tag{1}$$

is linear or not. If it is linear, then give its order.

Solution. This is not linear. □

Problem 4.1.3. *Determine if the differential equation*

$$5x \sin x = 3y^{(4)} - 4xy''' + \frac{5}{x}y'' - 2y' + x^3 y \tag{2}$$

is linear or not. If it is linear, then give its order.

Solution. This is a linear fourth-order differential equation. □

Problem 4.1.10. *Show that $\{1/x, x\}$ is a fundamental set of solutions to the homogeneous differential equation*

$$x^2 y'' + xy' - y = 0. \tag{3}$$

Find the general solution and find the solution with initial values $y(1) = 0$ and $y'(1) = -1$.

Solution. It is easily verified that both $y = 1/x$ and $y = x$ solve (3). To see that $1/x$ and x are linearly independent, note that

$$w(1/x, x)_{x=1} = \begin{vmatrix} 1/x & x \\ -1/x^2 & 1 \end{vmatrix}_{x=1} = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2 \neq 0.$$

Since the solution space of (3) is two-dimensional, we see that $\{1/x, x\}$ is a fundamental set of solutions to (3). The general solution is then given by

$$y = c_1 \frac{1}{x} + c_2 x.$$

Imposing the initial conditions $y(1) = 0$ and $y'(1) = -1$ gives the solution

$$y = \frac{1}{2}\frac{1}{x} - \frac{1}{2}x.$$

□

Problem 4.1.11. *Show that $\left\{e^x, e^{2x}, e^{-3x}\right\}$ is a fundamental set of solutions to the homogeneous differential equation*

$$y''' - 7y' + 6y = 0. \tag{4}$$

Find the general solution and find the solution with initial values $y(0) = 1$, $y'(0) = 0$, and $y''(0) = 0$.

Solution. It is easily verified that each of $y = e^x$, $y = e^{2x}$, and $y = e^{-3x}$ solves (4). To see that e^x, e^{2x}, e^{-3x} are linearly independent, note that

$$w\left(e^x, e^{2x}, e^{-3x}\right)_{x=0} = \begin{vmatrix} e^x & e^{2x} & e^{-3x} \\ e^x & 2e^{2x} & -3e^{-3x} \\ e^x & 4e^{2x} & 9e^{-3x} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 1 & 4 & 9 \end{vmatrix} = 20 \neq 0.$$

Since the solution space of (4) is three-dimensional, wee see that $\left\{e^x, e^{2x}, e^{-3x}\right\}$ is a fundamental set of solutions to (4). The general solution is then

$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{-3x}.$$

Imposing $y(0) = 1$, $y'(0) = 0$, and $y''(0) = 0$ gives

$$y = \frac{3}{2}e^x - \frac{3}{5}e^{2x} + \frac{1}{10}e^{-3x}.$$ □

Problem 4.1.15. *Show that $y_p = e^{-x}$ is a particular solution to the differential equation*

$$y''' - 7y' + 6y = 12e^{-x}. \tag{5}$$

Use Problem 4.1.11 to find the solution to (5) *with initial conditions $y(0) = 1$ and $y'(0) = y''(0) = 0$*

Solution. It is easily verified that $y_p = e^{-x}$ solves (5). By Problem 4.1.11, the general solution to (5) is

$$y = y_p + y_H = e^{-x} + c_1 e^x + c_2 e^{2x} + c_3 e^{-3x}.$$

Imposing $y(0) = 1$ and $y'(0) = y''(0) = 0$ gives

$$y = e^{-x} + \frac{1}{5}e^{2x} - \frac{1}{5}e^{-3x}.$$ □

Problem 4.1.17. *Show that $\left\{x, x^2\right\}$ is a fundamental set of solutions to*

$$\frac{1}{2}x^2 y'' - xy' + y = g(x). \tag{6}$$

Then, determine $g(x)$ so that $y_p = e^x$ is a particular solution to (6)*. Finally, determine the solution to* (6) *with initial values $y(1) = 1$ and $y'(1) = 0$.*

Solution. It is easily verified that $y = x$ and $y = x^2$ solve the homogeneous part of (6). Furthermore, x and x^2 are independent since

$$w\left(x, x^2\right)_{x=1} = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix}_{x=1} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1 \neq 0.$$

Since the homogeneous part of (6) has two-dimensional solution space, we see that $\left\{x, x^2\right\}$ is a fundamental set of solutions to (6).

Next, to determine $g(x)$ so that $y_p = e^x$ is a particular solution to (6), plug $y_p = e^x$ into (6) to obtain

$$g(x) = e^x\left(\frac{1}{2}x^2 - x + 1\right).$$

Finally, we see that the general solution to (6) is

$$y = y_p + y_H = e^x + c_1 x + c_2 x^2$$

and imposing $y(1) = 1$ and $y'(1) = 0$ gives

$$y = (2 - e)\,x - x^2 + e^x. \qquad \square$$