MATH 107.01
HOMEWORK #12 SOLUTIONS

Problem 4.2.2. *Determine the general solution to the differential equation*

$$y'' - y' - 6y = 0.$$

Solution. The characteristic polynomial is

$$p(\lambda) = \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2).$$

Corollary 4.7 in the book then implies that the general solution is

$$y = c_1 e^{3x} + c_2 e^{-2x}.$$ □

Problem 4.2.5. *Determine the general solution to the differential equation*

$$y''' + 4y'' - y' - 4y.$$

Solution. The characteristic polynomial is

$$p(\lambda) = \lambda^3 + 4\lambda^2 - \lambda - 4 = (x - 1)(x + 1)(x + 4).$$

Corollary 4.7 in the book then implies that the general solution is

$$y = c_1 e^{x} + c_2 e^{-x} + c_3 e^{-4x}.$$ □

Problem 4.2.7. *Determine the general solution to the differential equation*

$$y'' + 6y' + 9y = 0.$$

Solution. The characteristic polynomial is

$$p(\lambda) = \lambda^2 + 6\lambda + 9 = (x + 3)^2.$$

Theorem 4.8 in the book then implies that the general solution is

$$y = c_1 e^{-3x} + c_2 x e^{-3x}.$$ □

Problem 4.2.10. *Determine the general solution to the differential equation*

$$y^{(4)} + 2y''' + y'' = 0.$$

Solution. The characteristic polynomial is

$$p(\lambda) = \lambda^4 + 2\lambda^3 + \lambda^2 = \lambda^2 (\lambda + 1)^2.$$

Theorem 4.8 in the book then implies that the general solution is

$$y = c_1 + c_2 x + c_3 e^{-x} + c_4 x e^{-x}.$$ □

Problem 4.2.11. *Determine the general solution of the differential equation*

$$4y'' + 16y = 0.$$

Solution. The characteristic polynomial is

$$p(\lambda) = 4\lambda^2 + 16 = 4(\lambda - 2i)(\lambda + 2i).$$

Corollary 4.10 in the book then implies that the general solution is

$$y = c_1 \cos 2x + c_2 \sin 2x.$$ □

Problem 4.2.12. *Determine the general solution of the differential equation*

$$2y'' - 8y' + 14y = 0.$$

Solution. The characteristic polynomial is

$$p(\lambda) = \left(\lambda - \left(2 + i\sqrt{3}\right)\right)\left(\lambda - \left(2 - i\sqrt{3}\right)\right).$$

Corollary 4.10 in the book then implies that the general solution is

$$y = c_1 e^{2x} \cos 3x + c_2 e^{2x} \sin 3x.$$

□

Problem 4.2.13. *Determine the general solution of the differential equation*

$$y''' - 2y' + 4y = 0.$$

Solution. The characteric polynomial is

$$p(\lambda) = \lambda^3 - 2\lambda + 4\lambda = (\lambda + 2)\left(\lambda - (1 - i)\right)\left(\lambda - (1 + i)\right).$$

Corollary 4.10 in the book then implies that the general solution is

$$y = c_1 e^{-4x} + c_2 e^{x} \cos x + c_2 e^{x} \sin x.$$

□

Problem 4.2.20. *Consider*

$$4y'' + 5y' + y = 0. \tag{1}$$

(*a*) *Find the general solution to* (1).
(*b*) *Find the solution to* (1) *with initial conditions* $y(0) = 0$ *and* $y'(0) = -1$.

Solution. (*a*) The characteric polynomial is

$$p(\lambda) = 4\lambda^2 + 5\lambda + 1 = (\lambda + 1)\left(\lambda + \frac{1}{4}\right).$$

Corollary 4.7 in the book then implies that the general solution is

$$y = c_1 e^{-x} + c_2 e^{-x/4}.$$

(*b*) Imposing the initial conditions $y(0) = 0$ and $y'(0) = -1$ to the general solution gives the system

$$c_1 + c_2 = 0$$
$$-c_1 - \frac{1}{4} = -1$$

which has solution $c_1 = 4/3$ and $c_2 = -4/3$. Hence the solution is

$$y = \frac{4}{3}e^{-x} - \frac{4}{3}e^{-x/4}.$$

□

Problem 4.2.22. *Consider*

$$y''' - 5y'' = 0. \tag{2}$$

(*a*) *Find the general solution to* (2).
(*b*) *Find the solution to* (2) *with initial conditions* $y(-1) = 1$, $y'(-1) = 0$, *and* $y''(-1) = 2$.

Solution. (*a*) The characteristic polynomial is

$$p(\lambda) = \lambda^3 - 5\lambda^2 = \lambda^2 (\lambda - 5) .$$

Theorem 4.8 in the book then implies that the general solution is

$$y = c_1 + c_2 x + c_3 e^{5x}.$$

(*b*) Imposing the initial conditions $y(-1) = 1$, $y'(-1) = 0$, and $y''(-1) = 2$ gives the system

$$\begin{aligned} c_1 - c_2 + c_3 e^{-5} &= 1 \\ c_2 + 5c_3 e^{-5} &= 0 \\ 25c_3^{-5} &= 2 \end{aligned}$$

which has solution $c_1 = 13/25$, $c_2 = -2/5$, and $c_3 = 2e^5/25$. Hence the solution is

$$\frac{13}{25} - \frac{2}{5}x + \frac{2}{25}e^5 e^{5x}. \qquad \square$$

Problem 4.2.23. *Consider*

$$y''' - y'' - 6y' = 0. \tag{3}$$

(*a*) *Find the general solution to* (3).
(*b*) *Find the solution to* (3) *with initial conditions* $y(0) = 1$, $y'(0) = 0$, *and* $y''(0) = 2$.

Solution. (*a*) The characteristic polynomial is

$$p(\lambda) = \lambda^3 - \lambda^2 - 6\lambda = \lambda (\lambda + 2) (\lambda - 3) .$$

Corollary 4.7 in the book then implies that the general solution is

$$y = c_1 + c_2 e^{-2x} + c_3 e^{3x}.$$

(*b*) Imposing the initial conditions $y(0) = 1$, $y'(0) = 0$, and $y''(0) = 2$ gives the system

$$\begin{aligned} c_1 + c_2 + c_3 &= 1 \\ -2c_2 + 3c_3 &= 0 \\ 4c_2 + 9c_3 &= 2 \end{aligned}$$

which has solution $c_1 = 2/3$, $c_2 = 1/5$, and $c_3 = 2/15$. Hence the solution is

$$\frac{2}{3} + \frac{1}{5}e^{-2x} + \frac{2}{15}e^{3x}. \qquad \square$$

Problem 4.2.24. *Consider*

$$y''' - 4y'' + 5y' = 0. \tag{4}$$

(*a*) *Find the general solution to* (4).
(*b*) *Find the solution to* (4) *with initial conditions* $y(1) = 1$, $y'(1) = 0$, *and* $y''(1) = 2$.

Solution. (*a*) The characteristic polynomial is

$$p(\lambda) = \lambda^3 - 4\lambda^2 + 5\lambda = \lambda\big(\lambda - (2 - i)\big)\big(\lambda - (2 + i)\big).$$

Corollary 4.10 in the book then implies that the general solution is

$$y = c_1 + c_2 e^{2x} \cos x + c_3 e^{2x} \sin x.$$

(b) Imposing the initial conditions $y(1) = 1$, $y'(1) = 0$, and $y''(1) = 2$ gives the system

$$\begin{aligned} c_1 + c_2 e^2 \cos 1 + c_3 e^2 \sin 1 &= 1 \\ c_2 e^2 (2\cos 1 - \sin 1) + c_3 e^2 (2\sin 1 + \cos 1) &= 0 \\ c_2 e^2 (3\cos 1 - 4\sin 1) + c_3 e^2 (3\sin 1 + 4\cos 1) &= 2 \end{aligned}$$

which has solution

$$\begin{aligned} c_1 &= \frac{7}{5} \\ c_2 &= -\frac{2}{5} e^{-2} (2\sin 1 + \cos 1) \\ c_3 &= \frac{2}{5} e^{-2} (2\cos 1 - \sin 1). \end{aligned}$$

Hence the solution is

$$y = \frac{7}{5} - \frac{2}{5}(2\sin 1 + \cos 1)\, e^{2x-2} \cos x + \frac{2}{5}(2\cos 1 - \sin 1)\, e^{2x-2} \sin x. \qquad \square$$

Problem 4.2.29. *Suppose that a homogeneous linear differential equation with constant coefficients has characteristic polynomial*

$$p(\lambda) = (2\lambda + 4)\left(\lambda^2 - 9\right)^2 \left(\lambda^2 + 9\right)^2.$$

Find the general solution.

Solution. We may write

$$p(\lambda) = 2(\lambda + 2)(\lambda + 3)^2 (\lambda - 3)^2 (\lambda + 3i)^2 (\lambda - 3i)^2.$$

Hence the general solution is

$$\begin{aligned} (5) \quad y = \left(c_1 + c_2 x + c_3 x^2\right) \cos 3x + \left(c_4 + c_5 x + c_6 x^2\right) \sin 3x + c_7 e^{-2x} \\ + (c_8 + c_9 x)\, e^{-3x} + (c_{10} + c_{11} x)\, e^{3x}. \quad \square \end{aligned}$$

Problem 4.2.30. *Suppose that a homogeneous linear differential equation with constant coefficients has characteristic polynomial*

$$p(\lambda) = \lambda^3 (\lambda + 4)^2 \left(2\lambda^2 + 4\lambda + 4\right)^2 \left(2\lambda^2 - 2\lambda + 5\right).$$

Find the general solution.

Solution. We may write

$$p(\lambda) = 2\lambda^3 (\lambda + 4)^2 \left(\lambda - (1 - i)\right)^2 \left(\lambda - (1 + i)\right)^2 \left(\lambda - \left(\frac{1}{2} - \frac{3}{2}i\right)\right)^2 \left(\lambda - \left(\frac{1}{2} + \frac{3}{2}i\right)\right)^2.$$

Hence the general solution is

$$\begin{aligned} (6) \quad y = c_1 + c_2 x + c_3 x^2 + c_4 e^{-4x} + c_5 x e^{-4x} + c_6 e^{-x} \cos x + c_7 e^{-x} \sin x \\ + c_8 x e^{-x} \cos x + c_9 x e^{-x} \sin x + c_{10} e^{x/2} \cos \frac{3}{2} x + c_{11} e^{x/2} \sin \frac{3}{2} x. \quad \square \end{aligned}$$