MATH 107.01
HOMEWORK #13 SOLUTIONS

Problem 4.3.1. *Determine the general solution of*

$$y'' - y' - 6y = 3e^{2x}. \tag{1}$$

Solution. The characteristic polynomial is

$$p(\lambda) = \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2)$$

so that the homogeneous solution is

$$y_H = c_1 e^{3x} + c_2 e^{-2x}.$$

By Theorem 4.11, the particular solution is of the form $y_p = A^{2x}$. Substituting y_p into (1) gives

$$4Ae^{2x} - 2Ae^{2x} - 6Ae^{2x} = 3e^{2x}$$

so that $A = -4/3$. Hence the general solution is

$$y = y_H + y_p = c_1 e^{3x} + c_2 e^{-2x} - \frac{4}{x} e^{2x}.$$ □

Problem 4.3.4. *Determine the general solution of*

$$2y'' + 3y' + y = \cos x. \tag{2}$$

Solution. The characteristic polynomial is

$$p(\lambda) = 2\lambda^2 + 3\lambda + 1 = (\lambda + 1)(2\lambda + 1)$$

so that the homogeneous solution is

$$y_H = c_1 e^{-x} + c_2 e^{-x/2}.$$

By Theorem 4.11, the particular solution is of the form $y_p = A\cos x + B\sin x$. Substituting y_p into (2) gives

$$(-A + 3B)\cos x + (-3A - B)\sin x = \cos x.$$

This gives the system

$$\begin{aligned} -A + 3B &= 1 \\ -3A - B &= 0 \end{aligned}$$

which has solution $A = 1/10$ and $B = -3/10$. Hence the general solution is

$$y = y_H + y_p = c_1 e^{-x} + c_2 e^{-x/2} + \frac{1}{10}\cos x - \frac{3}{10}\sin x.$$ □

Problem 4.3.9. *Determine the general solution of*

$$4y'' + 16y = 3\cos 2x. \tag{3}$$

Solution. The characteristic polynomial is

$$p(\lambda) = 4\lambda^2 + 16 = 4(\lambda - 2i)(\lambda + 2i)$$

so that the homogeneous solution is

$$y_H = c_1 \cos 2x + c_2 \sin 2x.$$

By Theorem 4.11, the particular solution is of the form

$$y_p = Ax \cos 2x + Bx \sin 2x.$$

Substituting y_p into (3) gives

$$16B \cos 2x - 16A \sin 2x = 3 \cos 2x.$$

This gives the system

$$\begin{aligned} 16B &= 3 \\ -16A &= 0 \end{aligned}$$

which has solution $A = 0$ and $B = 3/16$. Hence the general solution is

$$y = y_H + y_p = c_1 \cos 2x + c_2 \sin 2x + \frac{3}{16} x \sin 2x. \qquad \square$$

Problem 4.3.11. *Determine the general solution of*

$$y'' + 8y' = 2x^2 - 7x + 3. \tag{4}$$

Proof. The characteristic polynomial is

$$p(\lambda) = \lambda^2 + 8\lambda = \lambda(\lambda + 8)$$

so that the homogeneous solution is

$$y_H = c_1 + c_2 e^{-8x}.$$

By Theorem 4.11, the particular solution is of the form

$$y_p = Ax^3 + Bx^2 + Cx.$$

Substituting y_p into (4) gives

$$24Ax^2 + (6A + 16B)x + 2B + 8C = 2x^2 - 7x + 3.$$

This gives the system

$$\begin{aligned} 24A &= 2 \\ 6A + 16B &= -7 \\ 2B + 8C &= 3 \end{aligned}$$

which has solution $A = 1/12$, $B = -15/32$, and $C = 63/128$. Hence the general solution is

$$y = y_H + y_p = c_1 + c_2 e^{-8x} + \frac{1}{12}x^3 - \frac{15}{32}x^2 + \frac{63}{128}x. \qquad \square$$

Problem 4.3.18. *Solve the initial value problem*

$$\begin{aligned} y'' + 4y' + 4y &= 2x - \sin 3x \\ y(0) &= 0 \\ y'(0) &= -1. \end{aligned}$$

Solution. The characteristic polynomial is

$$p(\lambda) = \lambda^2 + 4\lambda + 4 = (\lambda + 2)^2$$

so that the homogeneous solution is

$$y_H = c_1 e^{-2x} + c_2 x e^{-2x}.$$

By Theorem 4.11, the particular solution is of the form

$$y_p = A + Bx + C\cos 3x + D\sin 3x.$$

Substituting y_p into the differential equation gives

$$4A + 4B + 4Bx + (-5C + 12D)\cos 3x + (-12C - 5D)\sin 3x = 2x - \sin 3x.$$

This gives the system

$$\begin{aligned} 4A + 4B &= 0 \\ 4B &= 2 \\ -5C + 12D &= 0 \\ -12C - 5D &= -1 \end{aligned}$$

which has solution $A = -1/2$, $B = 1/2$, $C = 12/169$, and $D = 5/169$. So, the general solution is

$$y = y_H + y_p = c_1 e^{-2x} + c_2 x e^{-2x} - \frac{1}{2} + \frac{1}{2}x + \frac{12}{169}\cos 3x + \frac{5}{169}\sin 3x.$$

Now, imposing the initial conditions gives the system

$$\begin{aligned} c_1 &= \frac{145}{338} \\ -2c_1 + c_2 &= -\frac{537}{338} \end{aligned}$$

which has solution $c_1 = 145/338$ and $c_2 = -19/26$. Hence the solution is

$$y = \frac{145}{338}e^{-2x} - \frac{19}{26}xe^{-2x} - \frac{1}{2} + \frac{1}{2}x + \frac{12}{169}\cos 3x + \frac{5}{169}\sin 3x. \qquad \square$$

Problem 4.3.26. *Determine a form fot eh particular solution of*

$$y^{(4)} + 8y'' + 16y = 4x - 2e^{4x} - \cos 4x + 2x\sin 2x.$$

Proof. The characteristic polynomial is

$$p(\lambda) = \lambda^4 + 8\lambda^2 + 16 = (\lambda - 2i)^2(\lambda + 2i)^2.$$

Following Theorem 4.11, the particular solution is then of the form

$$y_p = A + Bx + Ce^{4x} + D\cos 4x + E\sin 4x + x^2(Fx + G)\cos 2x + x^2(Hx + I)\sin 2x. \qquad \square$$