MATH 107.01
HOMEWORK #14 SOLUTIONS

Problem 3.6.7. *A* 100*-gal tank contains* 40 *gal of an alcohol-water solution* 2 *gal of which is alcohol. A solution containing* 0.5 *gal alcohol/gal runs into the tank at a rate of* 3 *gal/min and the well-stirred mixture leaves the tank at a rate of* 2 *gal/min.*

(a) *How many gallons of alcohol are in the tank after* 10 *minutes?*
(b) *How many gallons of aocohol are in the tank when it overflows?*

Solution. Let $A(t)$ be the amount of alcohol in the tank at time t.

Note that the volume of the tank is given by $V(t) = 40 + t$ and that

$$A'(t) = \frac{1}{2} \cdot 3 - \frac{2}{V(t)} A(t) = \frac{3}{2} - \frac{2}{40+t} A(t).$$

This gives the initial value problem

$$A'(t) + \frac{2}{40+t} A(t) = \frac{3}{2}$$
$$A(0) = 2.$$

The integrating factor of this initial value problem is

$$I(t) = e^{\int \frac{2}{40+t}\, dt} = e^{\ln\left((40+t)^2\right)} = (40+t)^2.$$

The integrating factor theorem then gives

$$A(t) = \frac{1}{(40+t)^2} \int \frac{3}{2} (40+t)^2 \, dt$$
$$= \frac{3}{2(40+t)^2} \left\{ \frac{(40+t)^3}{3} + C \right\}.$$

Imposing the initial condition gives $C = -19200$ so that

$$A(t) = \frac{3}{2(40+t)^2} \left\{ \frac{(40+t)^3}{3} - 19200 \right\}.$$

(a) $A(10) = 337/25 = 13.48$
(b) The tank overfills at time $t = 60$. This gives $A(60) = 1178/25 = 47.12$. □

Problem 4.5.5. *A* 8*-lb object stretches a spring* 4 *ft. The mass-spring system is placed on a horizontal track that has a friction constant of* 1 *lb-sec/ft. The object is pulled 6 inches from equilibrium and released. Determine the motion of the object.*

Solution. The setup gives

$$m = \frac{8}{g} = \frac{8}{32} = \frac{1}{4}$$
$$k = \frac{mg}{L} = \frac{32}{4(4)} = 2$$

$$f = 1$$
$$s(0) = \frac{1}{2}$$
$$s'(0) = 0.$$

The initial value problem is then

$$\frac{1}{4}s'' + s' + 2s = 0$$
$$s(0) = \frac{1}{2}$$
$$s'(0) = 0.$$

The characteristic polynomial is

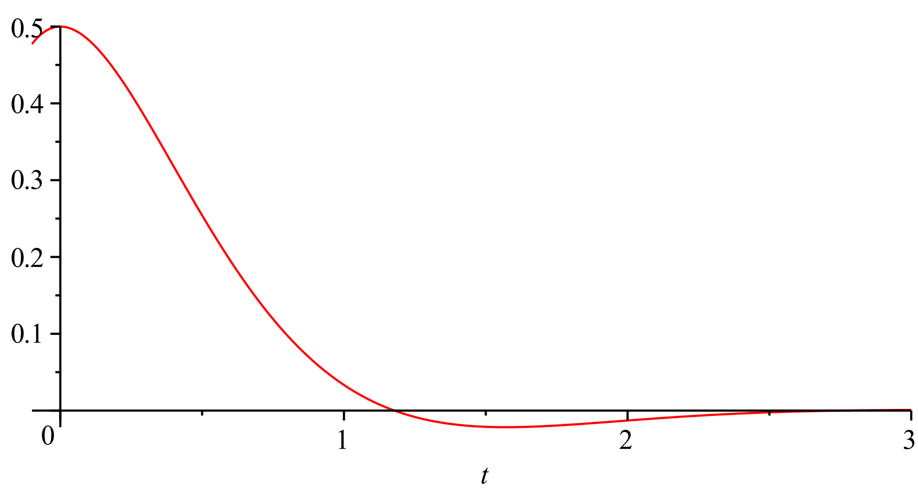
$$p(\lambda) = \frac{1}{4}\lambda^2 + \lambda + 2 = \frac{1}{4}\big(\lambda + (2 + 2i)\big)\big(\lambda + (2 - 2i)\big).$$

The solution is then

$$s(t) = e^{-2t}\left(c_1 \cos 2t + c_2 \sin 2t\right).$$

Imposing the initial conditions gives $c_1 = c_2 = 1/2$. Hence the solution is

$$s(t) = \frac{1}{2}e^{-2t}\left(\cos 2t + \sin 2t\right).$$ □



Problem 4.5.6. *A 5-kg mass stretches a spring 25 cm. The mass-spring system is hung vertically in a tall tank filled with oil offering a resistance to the motion of 40 kg/sec. The object is pulled 25 cm from rest and given an initial velocity of 50 cm/sec. Determine the motion of the object.*

Solution. The setup gives

$$\begin{aligned} m &= 5 \\ k &= \frac{mg}{L} = \frac{(5)(10)}{1/4} = 200 \\ f &= 40 \\ s(0) &= 25 \\ s'(0) &= 0. \end{aligned}$$

The initial problem is then

$$\begin{aligned} 5s'' + 40s' + 200s &= 0 \\ s(0) &= 25 \\ s'(0) &= 0. \end{aligned}$$

The characteristic polynomial is

$$p(\lambda) = 5\lambda^2 + 40\lambda + 200$$

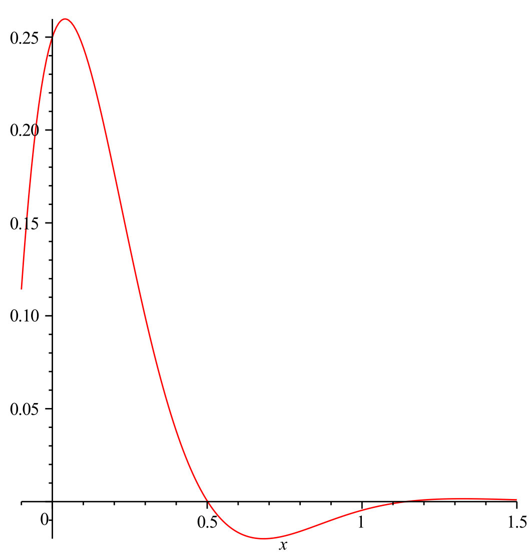
which has roots $\lambda = -4 + 2i\sqrt{6}$ and $\lambda = -4 + 2i\sqrt{6}$. The solution is then

$$s(t) = e^{-4t}\left(c_1 \cos 2\sqrt{6}t + c_2 \sin 2\sqrt{6}t\right).$$

Imposing the initial conditions gives $c_1 = 1/4$ and $c_2 = \sqrt{6}/8$. Hence the solution is

$$s(t) = e^{-4t}\left(\frac{1}{4}\cos 2\sqrt{6}t + \frac{\sqrt{6}}{8}\sin 2\sqrt{6}t\right).$$

□



Problem 4.5.7. *A 2-kg mass stretches a spring 1 m. This mass is hung vertically on the spring and then a shock absorber is attached that exerts a resistance of 14 kg/sec to motion. Determine the motion of the mass if it is pulled down 3 m and then released.*

Solution. The setup gives

$$\begin{aligned} m &= 2 \\ k &= \frac{mg}{L} = \frac{2\,(10)}{1} = 20 \\ f &= 14 \\ s(0) &= 3 \\ s'(0) &= 0. \end{aligned}$$

The initial value problem is then

$$\begin{aligned} 2s'' + 14s' + 20s &= 0 \\ s(0) &= 3 \\ s'(0) &= 0. \end{aligned}$$

The characteristic polynomial is

$$p(\lambda) = 2\lambda^2 + 14\lambda + 20 = 2\,(\lambda + 2)\,(\lambda + 5)\,.$$

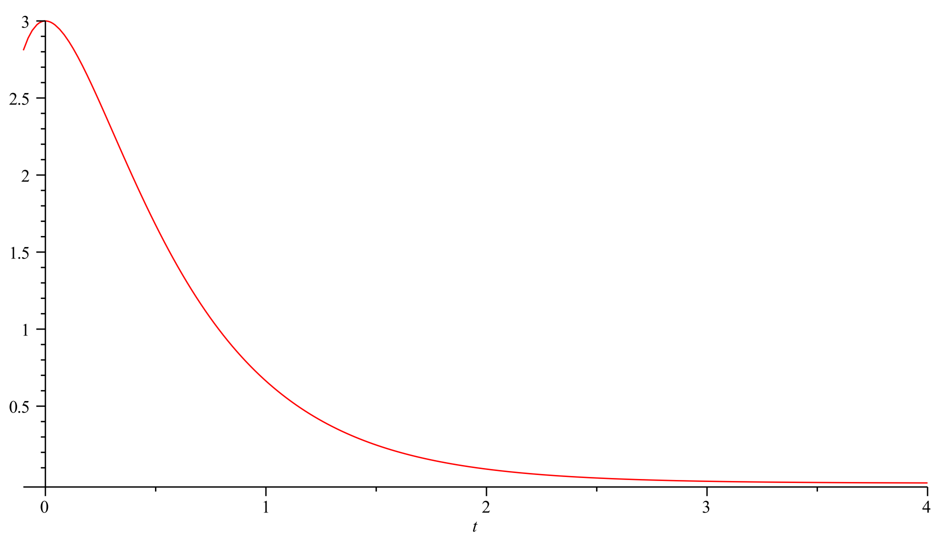
The solution is then

$$s(t) = c_1 e^{-2t} + c_2 e^{-5t}.$$

Imposing the initial conditions gives $c_1 = 5$ and $c_2 = -2$. Hence the solution is

$$s(t) = 5e^{-2t} - 2e^{-5t}.$$

□



Problem 4.5.8. *A 3-lb object stretches a spring 4 in. The mass-spring system is hung vertically and air offers a resistance to the motion of the object of 12 lb-sec/ft. The object is pushed up 3 inches from rest and given an initial velocity of 6 in/sec. Determine the motion of the object.*

Solution. The setup gives

$$\begin{aligned} m &= \frac{3}{32} \\ k &= \frac{mg}{L} = \frac{3}{4/12} = 9 \\ f &= 12 \\ s(0) &= -\frac{1}{4} \\ s'(0) &= \frac{1}{2}. \end{aligned}$$

This gives the initial value problem

$$\begin{aligned} \frac{3}{32}s'' + 12s' + 9s &= 0 \\ s(0) &= \frac{1}{4} \\ s'(0) &= -\frac{1}{2}. \end{aligned}$$

The characteristic polynomial is

$$p(\lambda) = \frac{3}{32}\lambda^2 + 12\lambda + 9 = \frac{3}{32}\left(\lambda - \left(64 - 20\sqrt{10}\right)\right)\left(\lambda + \left(64 + 20\sqrt{10}\right)\right).$$

The solution is then

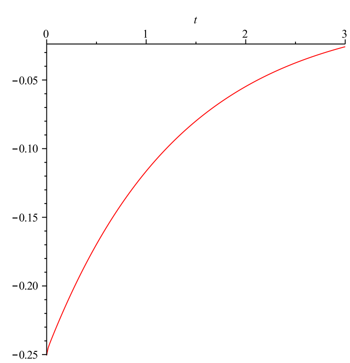
$$s(t) = c_1 e^{\left(-64+20\sqrt{10}\right)t} + c_2 e^{-\left(64+20\sqrt{10}\right)t}.$$

Imposing the initial conditions gives

$$c_1 = -\frac{31\sqrt{10}+100}{800}, \quad c_2 = \frac{31\sqrt{10}-100}{800}.$$

Hence the solution is

$$s(t) = -\frac{31\sqrt{10}+100}{800}e^{\left(-64+20\sqrt{10}\right)t} + \frac{31\sqrt{10}-100}{800}e^{-\left(64+20\sqrt{10}\right)t}.$$ □



Problem 4.5.11. *Solve Problem 4.5.5 when an external force of* $2\cos 2t$ *is applied to the system.*

Solution. This is the initial value problem

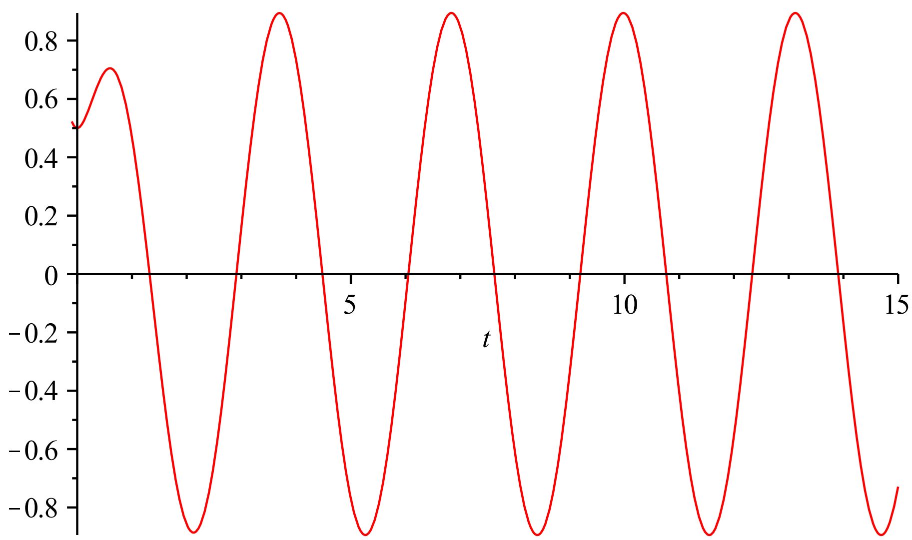
$$\frac{1}{4}s'' + x' + 2s = 2\cos 2t$$
$$s(0) = \frac{1}{2}$$
$$s'(0) = 0.$$

Since $\lambda = 2 + 2i$ is a root of the characteristic polynomial, the solution is of the form

$$s(t) = e^{-2t}\left(c_1 \cos 2t + c_2 \sin 2t\right) + A\cos 2t + B\sin 2t.$$

Imposing the initial conditions and comparing coefficients gives $c_1 = 1/10$, $c_2 = -7/10$, $A = 2/5$, and $B = 4/5$. Hence the solution is

$$s(t) = e^{-2t}\left(\frac{1}{10}\cos 2t - \frac{7}{10}\sin 2t\right) + \frac{2}{5}\cos 2t + \frac{4}{5}\sin 2t. \qquad \square$$



Problem 4.5.13. *Solve Problem 4.5.7 when an external force of* $6e^{-2t}$ *newtons is applied to the mass-spring system.*

Solution. This is the initial value problem

$$2s'' + 14s' + 20s = 6e^{-2t}$$
$$s(0) = 3$$
$$s'(0) = 0.$$

Since $\lambda = -2$ is a root of the characteristic polynomial, the solution is of the form

$$s(t) = c_1e^{-2t} + c_2e^{-5t} + Ate^{-2t}.$$

Imposing the initial conditions and comparing coefficients gives $c_1 = 14/3$, $c_2 = -5/3$, and $A = 1$. Hence the solution is

$$s(t) = \frac{14}{3}e^{-2t} - \frac{5}{3}e^{-5t} + te^{-2t}.$$ □