MATH 107.01
HOMEWORK #15 SOLUTIONS

Problem 5.1.3. *Determine if $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by*

$$T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y+z \\ z-y-x \\ xyz \end{bmatrix}$$

is linear.

Solution. Note that

$$T \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + T \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} = T \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = T \left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right).$$

Hence T is not linear. □

Problem 5.1.4. *Determine if $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by*

$$T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x-2y+5z \\ x+2z \end{bmatrix}$$

is linear.

Solution. This is $T(X) = AX$ where

$$A = \begin{bmatrix} 2 & -2 & 5 \\ 1 & 0 & 2 \end{bmatrix}.$$

Hence T is linear. □

Problem 5.1.7. *Determine if $T : P_2 \to P_2$ given by*

$$T\big(ax^2 + bx + c\big) = a\,(x+1)^2 + b\,(x+1) + c$$

is linear.

Solution. Note that

$$\begin{aligned}
T\Big(\lambda_1 \left(a_1 x^2 + b_1 x + c_1\right) &+ \lambda_2 \left(a_2 x^2 + b_2 + c_2\right)\Big) \\
&= T\Big(\left(\lambda_1 a_1 + \lambda_2 a_2\right) x^2 + \left(\lambda_1 b_1 + \lambda_2 b_2\right) x + \left(\lambda_1 c_1 + \lambda_2 c_2\right)\Big) \\
&= \left(\lambda_1 a_1 + \lambda_2 a_2\right)(x+1)^2 + \left(\lambda_1 b_1 + \lambda_2 b_2\right)(x+1) + \left(\lambda_1 c_1 + \lambda_2 c_2\right) \\
&= \lambda_1 a_1 (x+1)^2 + \lambda_1 b_1 (x+1) + \lambda_1 c_1 \\
&\qquad + \lambda_2 a_2 (x+1)^2 + \lambda_2 b_2 (x+1) + \lambda_2 c_2 \\
&= \lambda_1 \Big(a_1 (x+1)^2 + b_1 (x+1) + c_1\Big) \\
&\qquad + \lambda_2 \Big(a_2 (x+1)^2 + b_2 (x+1) + c_2\Big) \\
&= \lambda_1 T\big(a_1 x^2 + b_1 x + c_1\big) + \lambda_2 T\big(a_2 x^2 + b_2 x + c_2\big)
\end{aligned}$$

whenever $\lambda_1, \lambda_2 \in \mathbb{R}$ and $a_1x^2+b_1x+c_1, a_2x^2+b_2+c_2 \in P_2$. Hence T is linear. □

Problem 5.1.12. *Determine if $T : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ given by $T(A) = \det(A)$ is linear.*

Proof. Note that

$$T(I+I) = T(2I) = \det(2I) = 2^n \det(I) = 2^n \neq 2 = \det(I)+\det(I) = T(I)+T(I)$$

whenever $n \geq 2$. Hence T is not linear. □

Problem 5.1.13. *Recall that the collection $\mathbb{R}^+$ of positive real numbers is a vector space under the addition $x \oplus y = xy$ and the scalar multiplication $\lambda \odot x = x^\lambda$.*

(*a*) *Show that the natural logarithm is a linear transformation $\ln : \mathbb{R}^+ \to \mathbb{R}$.*
(*b*) *Show that the exponential map is a linear transformation $e : \mathbb{R} \to \mathbb{R}^+$.*

Solution. (*a*) Note that

$$\begin{aligned} \ln(\lambda_1 \odot x_1 \oplus \lambda_2 \odot x_2) &= \ln\left(x_1^{\lambda_1} \oplus x_2^{\lambda_2}\right) = \ln\left(x_1^{\lambda_1}x_2^{\lambda_2}\right) = \ln\left(x_1^{\lambda_1}\right) + \ln\left(x_2^{\lambda_2}\right) \\ &= \lambda_1 \ln(x_1) + \lambda_2 \ln(x_2) \end{aligned}$$

whenever $\lambda_1, \lambda_2 \in \mathbb{R}$ and $x_1, x_2 \in \mathbb{R}^+$. Hence ln is linear.

(*b*) Note that

$$e^{\lambda_1x_1+\lambda_2x_2} = e^{\lambda_1x_1}e^{\lambda_2x_2} = e^{\lambda_1x_1} \oplus e^{\lambda_2x_2} = (e^{x_1})^{\lambda_1} \oplus (e^{x_2})^{\lambda_2} = \lambda_1 \odot e^{x_1} \oplus \lambda_2 \odot e^{x_2}$$

whenever $\lambda_1, \lambda_2 \in \mathbb{R}$ and $x_1, x_2 \in \mathbb{R}$. Hence e is linear. □

Problem 5.1.18. *Find a matrix A that expresses the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ given by*

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 + 3x_3 - x_4 \\ 2x_1 + 3x_2 - x_3 - 2x_4 \\ 3x_1 + 7x_2 - 5x_3 - 3x_4 \end{bmatrix}$$

in the form of a matrix transformation $T(X) = AX$.

Solution. Here, $A \in M_{3\times 4}(\mathbb{R})$ satisfies

$$Ae_1 = Te_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad Ae_2 = Te_2 = \begin{bmatrix} -1 \\ 3 \\ 7 \end{bmatrix}$$

$$Ae_3 = Te_3 = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix} \qquad Ae_4 = Te_4 = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}.$$

This gives

$$A = \begin{bmatrix} 1 & -1 & 3 & -1 \\ 2 & 3 & -1 & -2 \\ 3 & 7 & -5 & -3 \end{bmatrix}.$$

□

Problem 5.1.20. *Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be a linear transformation such that*

$$T\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad T\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad T\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}.$$

(*b*) *Find* $T\begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

Solution. (*b*) First, note that

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

form a basis for $\mathbb{R}^3$. Furthermore, since

$$\text{rref}\left[\begin{array}{rrr|} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & x/2 - y/2 - z/2 \\ 0 & 1 & 0 & x/2 + y/2 + z/2 \\ 0 & 0 & 1 & x/2 + y/2 - z/2 \end{array}\right],$$

we have

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{2}(x - y - z)\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \frac{1}{2}(x + y + z)\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{2}(x + y - z)\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}. \tag{1}$$

Applying T to (1) then gives

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{2}(x - y - z)\,T\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \frac{1}{2}(x + y + z)\,T\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{2}(x + y - z)\,T\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} =$$

so that

$$\begin{aligned} T\begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \frac{1}{2}(x - y - z)\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + \frac{1}{2}(x + y + z)\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2}(x + y - z)\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 2x + y \\ x/2 + y/2 + x/2 \\ -x/2 + y/2 + z/2 \\ -x/2 - y/2 + z/2 \end{bmatrix}. \end{aligned}$$

□