MATH 107.01
HOMEWORK #16 SOLUTIONS

Problem 5.2.6. *Let $S, T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformations*

$$S\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}2x-y\\x+2y\end{bmatrix}, \quad T\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x+3y\\x-y\end{bmatrix}.$$

Find

$$TS\begin{bmatrix}x\\y\end{bmatrix}.$$

Solution. Compute

$$TS\begin{bmatrix}x\\y\end{bmatrix} = T\begin{bmatrix}2x-y\\x+2y\end{bmatrix} = \begin{bmatrix}(2x-y)+3(x+2y)\\(2x-y)-(x+2y)\end{bmatrix} = \begin{bmatrix}5x-5y\\x-3y\end{bmatrix}.$$ □

Problem 5.2.20. *Let $T : V \to V$ be a linear transformation.*

(a) *Let $v \in V$ and let $u \in \ker(T)$. Show that $T(u+v) = T(v)$.*
(b) *Let $u, v \in V$ such that $T(u) = T(v)$. Show that $u - v \in \ker(T)$.*

Solution. (a) Since $u \in \ker(T)$, we have $T(u) = \mathbf{0}$. Since T is a linear transformation, it follows that $T(u+v) = T(u) + T(v) = \mathbf{0} + T(v) = T(v)$.

(b) Note that $T(u-v) = T(u) - T(v) = T(u) - T(u) = \mathbf{0}$. Since $\ker(T) = \{x \in V : T(x) = \mathbf{0}\}$, it follows that $u - v \in \ker(T)$. □

Problem 5.2.23. *Let $S, T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformations given by*

$$S\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}2x-y\\x+2y\end{bmatrix}, \quad T\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x+3y\\x-y\end{bmatrix}.$$

(a) *Find matrices A and B so that T and S are expressed as the matrix transformations $T(X) = AX$ and $S(X) = BX$.*
(b) *Find the matrix C so that the composite ST is expressed in the form $ST(X) = CX$. Then verify that $C = BA$.*
(c) *Find the matrix D so that the composite TS is expressed in the form $TS(X) = DX$. Then verify that $D = AB$.*

Solution. (a) Note that

$$T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} \qquad S\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$$

$$T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}3\\-1\end{bmatrix} \qquad S\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-1\\2\end{bmatrix}$$

So, T and S are given by $T(X) = AX$ and $S(X) = BX$ where

$$A = \begin{bmatrix}1 & 3\\1 & -1\end{bmatrix}, \quad B = \begin{bmatrix}2 & -1\\1 & 2\end{bmatrix}.$$

(b) Note that ST is given by

$$ST\begin{bmatrix}x\\y\end{bmatrix} = S\begin{bmatrix}x+3y\\x-y\end{bmatrix} = \begin{bmatrix}2(x+3y)-(x-y)\\(x+3y)+2(x-y)\end{bmatrix} = \begin{bmatrix}x+7y\\3x+y\end{bmatrix}.$$

This gives

$$ST\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix}, \quad ST\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}7\\1\end{bmatrix}.$$

Hence ST is given by $ST(X) = CX$ where

$$C = \begin{bmatrix}1 & 7\\3 & 1\end{bmatrix}.$$

(c) By Problem 5.1.6, TS is given by

$$TS\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}5x-5y\\x-3y\end{bmatrix}.$$

This gives

$$TS\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}5\\1\end{bmatrix}, \quad TS\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-5\\-3\end{bmatrix}.$$

Hence TS is given by $TS(X) = DX$ where

$$D = \begin{bmatrix}5 & -5\\1 & -3\end{bmatrix}.$$

□