MATH 107.01
HOMEWORK #17 SOLUTIONS

Problem 5.3.1. *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation*

$$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \end{bmatrix}.$$

(*a*) *Find $[T]_\alpha^\alpha$ where α is the standard basis of $\mathbb{R}^2$.*
(*b*) *Let β be the basis for $\mathbb{R}^2$ consisting of*

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Find the change of basis matrix from α to β.
(*c*) *Find the change of basis matrix from β to α.*
(*d*) *Find $[T]_\beta^\beta$.*
(*e*) *Find $[v]_\beta$ where*

$$v = \begin{bmatrix} -2 \\ 3 \end{bmatrix}.$$

(*f*) *Find $\big[T(v)\big]_\beta$.*
(*g*) *Use (f) to find $T(v)$.*

Solution. (*a*) Note that

$$Te_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = e_1 + e_2$$
$$Te_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = e_1 - e_2.$$

This gives

$$[T]_\alpha^\alpha = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

(*b*) Let $\mathbf{1} : \mathbb{R}^2 \to \mathbb{R}^2$ be the identity transformation. Then

$$\mathbf{1}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = e_1 - e_2$$
$$\mathbf{1}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} = -2e_1 + e_2.$$

So, the change of basis matrix from α to β is

$$[\mathbf{1}]_\beta^\alpha = \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}.$$

(*c*) The change of basis matrix from β to α is

$$[\mathbf{1}]_\alpha^\beta = \left([\mathbf{1}]_\beta^\alpha\right)^{-1} = \begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix}.$$

(*d*) Compute

$$[T]_\beta^\beta = [\mathbf{1}]_\alpha^\beta \, [T]_\alpha^\alpha \, [\mathbf{1}]_\beta^\alpha = \begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -4 & 7 \\ -2 & 4 \end{bmatrix}.$$

(*e*) Here,

$$[v]_\beta = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

where c_1 and c_2 satisfy

$$c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}.$$

Since

$$\operatorname{rref} \left[\begin{array}{cc|c} 1 & -2 & -2 \\ -1 & 1 & 3 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & -4 \\ 0 & 1 & -1 \end{array}\right],$$

we have $c_1 = -4$ and $c_2 = -1$. Hence

$$[v]_\beta = \begin{bmatrix} -4 \\ -1 \end{bmatrix}.$$

(*f*) Compute

$$\big[T(v)\big]_\beta = [T]_\beta^\beta \, [v]_\beta = \begin{bmatrix} -4 & 7 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} -4 \\ -1 \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \end{bmatrix}.$$

(*g*) From (*f*), we have

$$T(v) = 9 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 4 \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}.$$ □

Problem 5.3.5. *Let $T : P_2 \to P_2$ be the linear transformation*

$$T\big(ax^2 + bx + c\big) = a\,(x+1)^2 + b\,(x+1) + c.$$

- (*a*) *Find $[T]_\alpha^\alpha$ where α is the standard basis $x^2, x, 1$ for P_2.*
- (*b*) *Let β be the basis $x^2 - 1, x^2 + 1, x + 1$ for P_2. Find the change of basis matrix from α to β.*
- (*c*) *Find the change of basis matrix from β to α.*
- (*d*) *Find $[T]_\beta^\beta$.*
- (*e*) *Find $[v]_\beta$ where $v = x^2 + x + 1$.*
- (*f*) *Find $\big[T(v)\big]_\beta$.*
- (*g*) *Use (f) to find $T(v)$.*

Solution. (*a*) Note that

$$\begin{aligned} T\big(x^2\big) &= (x+1)^2 = x^2 + 2x + 1 \\ T(x) &= x + 1 \\ T(1) &= 1. \end{aligned}$$

This gives

$$[T]_\alpha^\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

(*b*) Let $\mathbf{1} : P_2 \to P_2$ be the identity transformation. Then

$$\mathbf{1}\big(x^2 - 1\big) = x^2 - 1$$

$$\mathbf{1}\big(x^2+1\big) = x^2+1$$
$$\mathbf{1}(x+1) = x+1.$$

So, the change of basis matrix from α to β is

$$[\mathbf{1}]_\beta^\alpha = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}.$$

(c) The change of basis matrix from β to α is

$$[\mathbf{1}]_\alpha^\beta = \left([\mathbf{1}]_\beta^\alpha\right)^{-1} = \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}.$$

(d) Compute

$$[T]_\beta^\beta = [\mathbf{1}]_\alpha^\beta \, [T]_\alpha^\alpha \, [\mathbf{1}]_\beta^\alpha = \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 3/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ 2 & 2 & 1 \end{bmatrix}.$$

(e) Here,

$$[v]_\beta = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

where c_1, c_2, and c_3 satisfy

$$c_1\left(x^2-1\right) + c_2\left(x^2+1\right) + c_3\left(x+1\right) = x^2+x+1.$$

This gives the system

$$\begin{aligned} c_1 + c_2 &= 1 \\ c_3 &= 1 \\ -c_1 + c_2 + c_3 &= 1. \end{aligned}$$

Since

$$\operatorname{rref}\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1 \end{array}\right],$$

the system has solution $c_1 = 1/2$, $c_2 = 1/2$, and $c_3 = 1$. Hence

$$[v]_\beta = \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix}.$$

(f) Compute

$$\big[T(v)\big]_\beta = [T]_\beta^\beta \, [v]_\beta = \begin{bmatrix} 3/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 3 \end{bmatrix}.$$

(g) By (f), we have

$$T(v) = \frac{1}{2}\left(x^2-1\right) + \frac{1}{2}\left(x^2+1\right) + 3\left(x+1\right) = x^2+3x+3. \qquad \square$$

Problem 5.3.9. *Let* v_1, v_2, v_3 *form a basis* α *for a vector space* V *and let* $T : V \to V$ *be a linear transformation*

$$T(v_1) = v_1 - v_2$$
$$T(v_2) = v_2 - v_3$$
$$T(v_3) = v_3 - v_1.$$

(*a*) *Find* $[T]_\alpha^\alpha$.
(*b*) *Find* $\big[T(v)\big]_\alpha$ *where* $v = v_1 - 2v_2 + 3v_3$.
(*c*) *Use* (*b*) *to find* $T(v)$ *in terms of* v_1, v_2, v_3.

Solution. (*a*) Here,

$$[T]_\alpha^\alpha = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

(*b*) Here,

$$\big[T(v)\big]_\alpha = [T]_\alpha^\alpha \, [v]_\alpha = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ 5 \end{bmatrix}.$$

(*c*) From (*b*), we have

$$T(v) = -2v_1 - 3v_2 + 5v_3.$$

□