MATH 107.01
HOMEWORK #18 SOLUTIONS

Problem 5.4.5. *Find the eigenvalues and bases for the eigenspaces of*

$$A = \begin{bmatrix} 0 & 3 \\ 4 & 0 \end{bmatrix}.$$

Solution. The characteristic polynomial is

$$p_A(\lambda) = \det(\lambda I - A) = \det \begin{bmatrix} \lambda & -3 \\ -4 & \lambda \end{bmatrix} = \lambda^2 - 12 = \left(\lambda - 2\sqrt{3}\right)\left(\lambda + 2\sqrt{3}\right).$$

The eigenvalues of A are then $\lambda_1 = 2\sqrt{3}$ and $\lambda_2 = -2\sqrt{3}$.

Next, note that

$$E_{2\sqrt{3}} = \text{null}\left(2\sqrt{3}I - A\right) = \text{null} \begin{bmatrix} 2\sqrt{3} & -3 \\ -4 & 2\sqrt{3} \end{bmatrix}.$$

Since

$$\text{rref} \left[\begin{array}{cc:c} 2\sqrt{3} & -3 & 0 \\ -4 & 2\sqrt{3} & 0 \end{array}\right] = \left[\begin{array}{cc:c} 1 & -\sqrt{3}/2 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

the solutions to $\left(2\sqrt{3}I - A\right) v = \mathbf{0}$ are of the form

$$v = x_2 \begin{bmatrix} \sqrt{3}/2 \\ 1 \end{bmatrix}.$$

This gives

$$E_{2\sqrt{3}} = \text{Span}\left\{ \begin{bmatrix} \sqrt{3}/2 \\ 1 \end{bmatrix} \right\}.$$

Now, note that

$$E_{-2\sqrt{3}} = \text{null}\left(-2\sqrt{3}I - A\right) = \text{null} \begin{bmatrix} -2\sqrt{3} & -3 \\ -4 & -2\sqrt{3} \end{bmatrix}.$$

Since

$$\text{rref} \left[\begin{array}{cc:c} -2\sqrt{3} & -3 & 0 \\ -4 & -2\sqrt{3} & 0 \end{array}\right] = \left[\begin{array}{cc:c} 1 & \sqrt{2}/3 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

the solutions to $\left(-2\sqrt{3}I - A\right) v = \mathbf{0}$ are of the form

$$v = x_2 \begin{bmatrix} -\sqrt{3}/2 \\ 1 \end{bmatrix}.$$

This gives

$$E_{-2\sqrt{3}} = \text{Span}\left\{ \begin{bmatrix} -\sqrt{3}/2 \\ 1 \end{bmatrix} \right\}.$$ □

Problem 5.4.8. *Find the eigenvalues and bases for the eigenspaces of*

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Solution. The characteristic polynomial is

$$p_A(\lambda) = \det(\lambda I - A) = \det \begin{bmatrix} \lambda - 1 & -1 & 0 \\ 0 & \lambda - 1 & 1 \\ 0 & 0 & \lambda - 2 \end{bmatrix} = (\lambda - 1)^2 (\lambda - 2).$$

The eigenvalues of A are then $\lambda_1 = 1$ and $\lambda_2 = 2$.

Next, note that

$$E_1 = \text{null}(I - A) = \text{null} \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

Since

$$\text{rref} \left[\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

the solutions to $(I - A)v = \mathbf{0}$ are of the form

$$v = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

This gives

$$E_1 = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Now, note that

$$E_2 = \text{null}(2I - A) = \text{null} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since

$$\text{rref} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

the solutions to $(2I - A)v = \mathbf{0}$ are of the form

$$v = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

This gives

$$E_2 = \text{Span}\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

□

Problem 5.4.9. *Find the eigenvalues and bases for the eigenspaces of*

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 1 & 4 \end{bmatrix}.$$

Proof. The characteristic polynomial is

$$p_A(\lambda) = \det(\lambda I - A) = \det \begin{bmatrix} \lambda - 4 & 0 & 0 \\ -1 & \lambda - 4 & 0 \\ 0 & -1 & \lambda - 4 \end{bmatrix} = (\lambda - 4)^3.$$

The eigenvalues of A are then $\lambda_1 = 4$.

Next, note that

$$E_4 = \operatorname{null}(4I - A) = \operatorname{null} \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

Since

$$\operatorname{rref} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

the solutions to $(4I - A)v = \mathbf{0}$ are of the form

$$v = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

This gives

$$E_4 = \operatorname{Span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

□

Problem 5.4.16. *Find the eigenvalues and bases for the eigenspaces of*

$$A = \begin{bmatrix} -4 & 5 \\ -4 & 4 \end{bmatrix}.$$

Solution. The characteristic polynomial is

$$\begin{aligned} p_A(\lambda) = \det(\lambda I - A) &= \det \begin{bmatrix} \lambda + 4 & -5 \\ 4 & \lambda - 4 \end{bmatrix} \\ &= (\lambda + 4)(\lambda - 4) + 20 = \lambda^2 - 16 + 20 = \lambda^2 + 4 \\ &= (\lambda + 2i)(\lambda - 2i). \end{aligned}$$

The eigenvalues of A are then $\lambda_1 = 2i$ and $\lambda_2 = -2i$.

Next, note that

$$E_{2i} = \operatorname{null}(2iI - A) = \operatorname{null} \begin{bmatrix} 2i + 4 & -5 \\ 4 & 2i - 4 \end{bmatrix}.$$

Since

$$\operatorname{rref} \left[\begin{array}{cc|c} 2i + 4 & -5 & 0 \\ 4 & 2i - 4 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & -1 + i/2 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

the solutions to $(2iI - A)v = \mathbf{0}$ are of the form

$$v = x_2 \begin{bmatrix} 1 - i/2 \\ 1 \end{bmatrix}.$$

This gives

$$E_{2i} = \operatorname{Span}\left\{ \begin{bmatrix} 1 - i/2 \\ 1 \end{bmatrix} \right\}.$$

Now, by Theorem 5.17,

$$E_{-2i} = \text{Span}\left\{\begin{bmatrix} 1+i/2 \\ 1 \end{bmatrix}\right\}. \qquad \square$$

Problem 5.4.17. *Find the eigenvalues and bases for the eigenspaces of*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

Solution. The characteristic polynomial is

$$\begin{aligned} p_A(\lambda) &= \det(\lambda I - A) \\ &= \det\begin{bmatrix} \lambda-1 & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & 1 & \lambda \end{bmatrix} \\ &= (\lambda-1)\det\begin{bmatrix} \lambda & -1 \\ 1 & \lambda \end{bmatrix} \\ &= (\lambda-1)\left(\lambda^2+1\right) \\ &= (\lambda-1)(\lambda-i)(\lambda+i). \end{aligned}$$

The eigenvalues of A are then $\lambda_1 = 1$, $\lambda_2 = i$, and $\lambda_3 = -i$.

Next, note that

$$E_1 = \text{null}(I-A) = \text{null}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Since

$$\text{rref}\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

the solutions to $(I-A)v = \mathbf{0}$ are of the form

$$v = x_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

This gives

$$E_1 = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}.$$

Now, note that

$$E_i = \text{null}(iI-A) = \text{null}\begin{bmatrix} i-1 & 0 & 0 \\ 0 & i & -1 \\ 0 & 1 & i \end{bmatrix}.$$

Since

$$\text{rref}\left[\begin{array}{ccc|c} i-1 & 0 & 0 & 0 \\ 0 & i & -1 & 0 \\ 0 & 1 & i & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

the solutions to $(iI - A)\, v = \mathbf{0}$ are of the form

$$v = x_3 \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}.$$

This gives

$$E_i = \operatorname{Span}\left\{ \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix} \right\}.$$

Finally, Theorem 5.17 gives

$$E_{-i} = \operatorname{Span}\left\{ \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix} \right\}.$$ □

Problem 5.4.20. *Find the eigenvalues and bases for the eigenspaces of* $D = \operatorname{diag}(d_1, d_2, \ldots, d_n)$.

Solution. The characteristic polynomial is

$$\begin{aligned} p_D(\lambda) &= \det(\lambda I - D) \\ &= \det\big(\lambda I - \operatorname{diag}(d_1, d_2, \ldots, d_n)\big) \\ &= \det\big(\operatorname{diag}(\lambda - d_1, \lambda - d_2, \ldots, \lambda - d_n)\big) \\ &= (\lambda - d_1)(\lambda - d_2)\cdots(\lambda - d_n). \end{aligned}$$

The (not necessarily distinct) eigenvalues of D are then $\lambda_1 = d_1, \lambda_2 = d_2, \ldots, \lambda_n = d_n$. Since $De_k = d_k = \lambda_k$, we have $E_{d_k} = \operatorname{Span}\{e_k\}$. □

Problem 5.4.26. *Let* $A \in M_{n\times n}(\mathbb{R})$*. Show that* A *and* $A^\top$ *have the same eigenvalues.*

Proof. Note that

$$\begin{aligned} p_A(\lambda) &= \det(\lambda I - A) = \det\left((\lambda I - A)^\top\right) = \det\left(\lambda I^\top - A^\top\right) \\ &= \det\left(\lambda I - A^\top\right) = p_{A^\top}(\lambda) \end{aligned}$$

so that A and $A^\top$ have the same characteristic polynomials. Hence A and $A^\top$ have the same eigenvalues. □