MATH 107.01
HOMEWORK #19 SOLUTIONS

Problem 5.5.5. *Diagonalize*

$$A = \begin{bmatrix} 0 & 3 \\ 4 & 0 \end{bmatrix}$$

if possible.

Solution. From Homework #18, the eigenvalues of A are $\lambda_1 = 2\sqrt{3}$ and $\lambda_2 = -2\sqrt{3}$. Furthermore, the eigenspaces of A are given by

$$E_{2\sqrt{3}} = \operatorname{Span}\left\{ \begin{bmatrix} \sqrt{3}/2 \\ 1 \end{bmatrix} \right\}, \quad E_{-2\sqrt{3}} = \operatorname{Span}\left\{ \begin{bmatrix} -\sqrt{3}/2 \\ 1 \end{bmatrix} \right\}.$$

It follows that $\dim\left(E_{2\sqrt{3}}\right) + \dim\left(E_{-2\sqrt{3}}\right) = 1 + 1 = 2$ so that A is diagonalizable. Finally, to diagonalize A, take

$$P = \begin{bmatrix} \sqrt{3}/2 & -\sqrt{3}/2 \\ 1 & 1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \sqrt{3}/2 & 0 \\ 0 & -\sqrt{3}/2 \end{bmatrix}.$$ □

Problem 5.5.8. *Diagonalize*

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

if possible.

Solution. From Homework #18, the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 2$. Furthermore, the eigenspaces of A are given by

$$E_1 = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}, \quad E_2 = \operatorname{Span}\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

It follows that $\dim(E_1) + \dim(E_2) = 1 + 1 = 2 \neq 3$. Hence A is not diagonalizable. □

Problem 5.5.9. *Diagonalize*

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 1 & 4 \end{bmatrix}$$

if possible.

Solution. From Homework #18, A has exactly one eigenvalue $\lambda = 4$. Furthermore, the eigenspace of A is given by

$$E_4 = \operatorname{Span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

It follows that $\dim(E_4) = 1 \neq 3$. Hence A is not diagonalizable. □

Problem 5.5.16. *Diagonalize*

$$A = \begin{bmatrix} -4 & 5 \\ -4 & 4 \end{bmatrix}$$

if possible.

Solution. From Homework #18, the eigenvalues of A are $\lambda_1 = 2i$ and $\lambda_2 = -2i$. Furthermore, the eigenspaces of A are given by

$$E_{2i} = \operatorname{Span}\left\{ \begin{bmatrix} 1 - i/2 \\ 1 \end{bmatrix} \right\}, \quad E_{-2i} = \operatorname{Span}\left\{ \begin{bmatrix} 1 + i/2 \\ 1 \end{bmatrix} \right\}.$$

It follows that $\dim(E_{2i}) + \dim(E_{-2i}) = 1 + 1 = 2$ so that A is diagonalizable. Finally, to diagonalize A, take

$$P = \begin{bmatrix} 1 - i/2 & 1 + i/2 \\ 1 & 1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix}. \qquad \square$$

Problem 5.5.17. *Diagonalize*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

if possible.

Solution. From Homework #18, the eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = i$, and $\lambda_3 = -i$. Furthermore, the eigenspaces of A are given by

$$E_1 = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}, \quad E_i = \operatorname{Span}\left\{ \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix} \right\}, \quad E_{-i} = \operatorname{Span}\left\{ \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix} \right\}.$$

It follows that $\dim(E_1) + \dim(E_i) + \dim(E_{-i}) = 1 + 1 + 1 = 3$ so that A is diagonalizable. Finally, to diagonalize A, take

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -i & i \\ 0 & 1 & 1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{bmatrix}. \qquad \square$$

Problem 5.5.31. *Let $A, B \in M_{n\times n}(\mathbb{R})$ be similar. Show that $p_A(\lambda) = p_B(\lambda)$.*

Solution. Since A and B are similar, there is a $P \in \mathrm{GL}_n(\mathbb{R})$ such that $A = P^{-1}BP$. It follows that

$$\begin{aligned} p_A(\lambda) &= \det(\lambda I - A) = \det\left(\lambda P^{-1}P - P^{-1}BP\right) = \det\left(P^{-1}(\lambda I - B)P\right) \\ &= \det\left(P^{-1}\right)\det(\lambda I - B)\det(P) = \det\left(P^{-1}\right)\det(P)p_B(\lambda) \\ &= \det\left(P^{-1}P\right)p_B(\lambda) = \det(I)p_B(\lambda) = p_B(\lambda). \end{aligned} \qquad \square$$

Problem 5.5.32. *Let $A, B, C \in M_{n\times n}(\mathbb{R})$.*

(*a*) *Show that A is similar to A.*
(*b*) *Show that A is similar to B if B is similar to A.*
(*c*) *Show that A is similar to C if A is similar to B and B is similar to C.*

Solution. (a) Note that $A = IAI = I^{-1}AI$ so that A is similar to A.

(b) Suppose that B is similar to A so that there exists a $P \in \mathrm{GL}_n(\mathbb{R})$ such that $B = P^{-1}AP$; let $Q = P^{-1}$. Then $A = PBP^{-1} = Q^{-1}BQ$. Hence A is similar to B.

(c) Suppose that A is similar to B and that B is similar to C so that there exist $P, Q \in \mathrm{GL}_n(\mathbb{R})$ such that $A = P^{-1}BP$ and $B = Q^{-1}CQ$; let $R = PQ$. Then $A = P^{-1}BP = Q^{-1}P^{-1}CPQ = (PQ)^{-1}C(PQ) = R^{-1}CR$. Hence A is similar to C. □

Problem 5.5.36. *Let $A \in M_{n\times n}(\mathbb{R})$ have distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. Show that A is diagonalizable.*

Proof. Since $\deg p_A(\lambda) = n$, each eigenvalue of A has multiplicity 1 in $p_A(\lambda)$. It follows that $\dim(E_{\lambda_1}) = \cdots = \dim(E_{\lambda_n}) = 1$. Hence

$$\dim(E_{\lambda_1}) + \cdots + \dim(E_{\lambda_n}) = n$$

so that A is diagonalizable. □

Problem 5.6.3. *Let $T : P_1 \to P_1$ be the linear transformation given by $T(ax+b) = 2bx + (a+b)$.*

(a) *Find the eigenvalues and bases for the eigenspaces of T.*
(b) *Determine whether T is diagonalizable.*

Solution. Let $\alpha = \{x, 1\}$ be the standard basis for P_1. Then $T(x) = 1$ and $T(1) = 2x + 1$ so that

$$[T]_\alpha^\alpha = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}.$$

Next, note that

$$p_{[T]_\alpha^\alpha}(\lambda) = \det(\lambda I - [T]_\alpha^\alpha) = \det \begin{bmatrix} \lambda & -2 \\ -1 & \lambda - 1 \end{bmatrix} = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$$

so that $[T]_\alpha^\alpha$ has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 2$. Now, the eigenspaces of $[T]_\alpha^\alpha$ are given by

$$E_{-1} = \mathrm{Span}\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}, \quad E_2 = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}.$$

It follows that $\dim(E_{-1}) + \dim(E_2) = 1 + 1 = 2$ so that $[T]_\alpha^\alpha$ is diagonalizable. Hence T has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 2$, the eigenspaces of T are

$$V_{-1} = \mathrm{Span}\{-2x + 1\}, \quad V_2 = \mathrm{Span}\{x + 1\},$$

and T is diagonalizable. □