2.1 Vector Spaces

<u>Def</u>: A <u>vector space</u> over $\mathbb{R}$ is a collection V with an operation of addition and scalar multiplication such that

1. $u+v = v+u$ for all $u, v \in V$
2. $u+(v+w) = (u+v)+w$ for all $u, v, w \in V$
3. there is an element $\vec{0} \in V$ such that $v + \vec{0} = v$ for all $v \in V$
4. for every $v \in V$, there is an element $-v \in V$ such that $v + (-v) = \vec{0}$
5. $\lambda(u+v) = \lambda u + \lambda v$ for all $\lambda \in \mathbb{R}$ and all $u, v \in V$
6. $(\lambda_1 + \lambda_2)u = \lambda_1 u + \lambda_2 u$ for all $\lambda_1, \lambda_2 \in \mathbb{R}$ and all $u \in V$
7. $\lambda_1(\lambda_2 u) = (\lambda_1 \lambda_2) u$ for all $\lambda_1, \lambda_2 \in \mathbb{R}$ and all $u \in V$
8. $1v = v$ for all $v \in V$

Elements of a vector space V are called <u>vectors</u>.

$\underline{Ex}$: $M_{m\times n}(\mathbb{R})$ is a vector space

① $A+B = B+A$ for all $A, B \in M_{m\times n}(\mathbb{R})$

Theorem 1.2 (1) ✓

② $A+(B+C) = (A+B)+C$ for all $A, B, C \in M_{m\times n}(\mathbb{R})$

Theorem 1.2 (2) ✓

③ there is an element $\vec{0} \in M_{m\times n}(\mathbb{R})$ such that $A + \vec{0} = A$ for all $A \in M_{m\times n}(\mathbb{R})$

Here, $\vec{0} = 0_{m\times n}$ ✓

④ for every $A \in M_{m\times n}(\mathbb{R})$, there is an element $-A \in M_{m\times n}(\mathbb{R})$ such that $A + (-A) = \vec{0}$

Here, $-A = (-1)A$ ✓

⑤ $\lambda(A+B) = \lambda A + \lambda B$ for all $\lambda \in \mathbb{R}$ and all $A, B \in M_{m\times n}(\mathbb{R})$

Theorem 1.2 (4) ✓

⑥ $(\lambda_1 + \lambda_2)A = \lambda_1 A + \lambda_2 A$ for all $\lambda_1, \lambda_2 \in \mathbb{R}$ and all $A \in M_{m\times n}(\mathbb{R})$

Theorem 1.2 (5) ✓

⑦ $\lambda_1(\lambda_2 A) = (\lambda_1\lambda_2)A$ for all $\lambda_1, \lambda_2 \in \mathbb{R}$ and all $A \in M_{m\times n}(\mathbb{R})$

Theorem 1.2 (3) ✓

⑧ $1 \cdot A = A$ for all $A \in M_{m\times n}(\mathbb{R})$ ✓ ∎

Ex: Let $\mathcal{C}^k(a,b)$ be the collection of all k-times differentiable functions on the open interval (a,b) with addition and scalar multiplication given by

$$(f+g)(x) = f(x) + g(x)$$
$$(\lambda f)(x) = \lambda f(x).$$

① $(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x)$ ✓

② $$\big(f+(g+h)\big)(x) = f(x) + (g+h)(x)$$
$$= f(x) + \big(g(x) + h(x)\big)$$
$$= \big(f(x) + g(x)\big) + h(x)$$
$$= (f+g)(x) + h(x)$$
$$= \big((f+g)+h\big)(x)$$ ✓

③ Here, $\vec{o}$ is the function $\vec{o}(x) = 0$:
$$(f+\vec{o})(x) = f(x) + \vec{o}(x) = f(x) + 0 = f(x)$$ ✓

④ Here, for $f \in \mathcal{C}^k(a,b)$, $-f$ is the function $(-f)(x) := -f(x)$:
$$\big(f+(-f)\big)(x) = f(x) + (-f)(x) = f(x) - f(x) = 0 = \vec{o}(x)$$ ✓

⑤-⑧ are also satisfied (Exercise)

So, $\mathcal{C}^k(a,b)$ is a vectorspace. ∎

Ex: $GL_n(\mathbb{R})$ is not a vector space. Addition is not well-defined:
For $A \in GL_n(\mathbb{R})$, $A + (-A) = O_{n \times n}$, but $O_{n\times n}$ is not invertible!

Ex: Let V be the collection of all pairs of real numbers (x,y) with addition and scalar multiplication defined by

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2 + 1, y_1 + y_2)$$

$$\lambda (x, y) = (\lambda x, \lambda y).$$

① $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2 + 1, y_1 + y_2)$

$= (x_2 + x_1 + 1, y_2 + y_1)$

$= (x_2, y_2) + (x_1, y_1)$ ✓

② $(x_1, y_1) + \big((x_2, y_2) + (x_3, y_3)\big)$

$= (x_1, y_1) + (x_2 + x_3 + 1, y_2 + y_3)$

$= (x_1 + x_2 + x_3 + 1 + 1, y_1 + y_2 + y_3)$

$= (x_1 + x_2 + 1, y_1 + y_2) + (x_3, y_3)$

$= \big((x_1, y_1) + (x_2, y_2)\big) + (x_3, y_3)$ ✓

③ Here, we have $\vec{0} = (-1, 0)$:

$$(x, y) + (-1, 0) = (x - 1 + 1, y + 0) = (x, y) \checkmark$$

Note that $\vec{0} \neq (0, 0)$!

④ Here, $-(x, y) = (-x - 2, -y)$:

$$\begin{aligned}(x, y) + (-(x, y)) &= (x, y) + (-x - 2, -y) \\ &= (x - x - 2 + 1, y - y) \\ &= (-1, 0) \\ &= \vec{0} \checkmark\end{aligned}$$

⑤ $\lambda((x_1, y_1) + (x_2, y_2))$

$$= \lambda(x_1 + x_2 + 1, y_1 + y_2)$$

$$= (\lambda x_1 + \lambda x_2 + \lambda, \lambda y_1 + \lambda y_2)$$

- but -

$\lambda(x_1, y_1) + \lambda(x_2, y_2)$

$$= (\lambda x_1, \lambda y_1) + (\lambda x_2, \lambda y_2)$$

$$= (\lambda x_1 + \lambda x_2 + 1, \lambda y_1 + \lambda y_2)$$

So, $\lambda((x_1, y_1) + (x_2, y_2)) \neq \lambda(x_1, y_1) + \lambda(x_2, y_2)$

Property ⑤ is not satisfied! V is not a vector space!

Additionally, ⑥ holds but ⑦ and ⑧ do not hold. ∎

Ex: Let $\mathbb{R}^+$ be the collection of all positive real numbers with addition and scalar multiplication defined by

$$x \oplus y = xy$$

$$\lambda \odot x = x^\lambda .$$

Here, we use $\oplus$ and $\odot$ to avoid confusion with usual addition and scalar multiplication. Then $\mathbb{R}^+$ is a vector space

1. $x \oplus y = xy = yx = y \oplus x$ ✓

2. $x \oplus (y \oplus z) = x(yz) = (xy)z = (x \oplus y) \oplus z$ ✓

3. Here, $\vec{0} = 1$: $x \oplus 1 = x \cdot 1 = x$ ✓

4. Here, ~~x~~ for $x \in \mathbb{R}^+$, $-x = \frac{1}{x}$:

$$x \oplus (-x) = x \oplus \tfrac{1}{x} = x\left(\tfrac{1}{x}\right) = 1 = \vec{0} \checkmark$$

5. $\lambda \odot (x \oplus y) = \lambda \odot (xy) = (xy)^\lambda = x^\lambda y^\lambda$

$$= x^\lambda \oplus y^\lambda = (\lambda \odot x) \oplus (\lambda \odot y) \checkmark$$

6. $(\lambda_1 + \lambda_2) \odot x = x^{\lambda_1 + \lambda_2} = x^{\lambda_1} x^{\lambda_2} = x^{\lambda_1} \oplus x^{\lambda_2}$

$$= (\lambda_1 \odot x) \oplus (\lambda_2 \odot x) \checkmark$$

7. $\lambda_1 \odot (\lambda_2 \odot x) = \lambda_1 \odot (x^{\lambda_2}) = (x^{\lambda_2})^{\lambda_1} = x^{\lambda_1 \lambda_2} = (\lambda_1 \lambda_2) \odot x$ ✓

8. $1 \odot x = x^1 = x$ ✓ ■

<u>Thm</u>: Let V be a vector space. Then

① $\vec{0}$ is unique

② $-v$ is unique for every $v \in V$.

<u>pf</u>: ① Suppose $\vec{0}$ and $\vec{0}'$ are two zero vectors in V. Then

$$\vec{0} = \vec{0} + \vec{0}' = \vec{0}'.$$

② Suppose $-v$ and $-v'$ are two negatives of v. then

$$-v = -v + \vec{0} = -v + (v - v') = (-v + v) - v'$$

$$= \vec{0} - v' = v'. \quad \blacksquare$$

<u>Thm</u>: Let V be a vector space. Then

① for every $v \in V$, $0 \cdot v = \vec{0}$

② for every $\lambda \in \mathbb{R}$, $\lambda \cdot \vec{0} = \vec{0}$

③ for every $v \in V$, $(-1) \cdot v = -v$.

<u>pf</u>: ① Note that

$$\vec{0} = 0 \cdot v - 0 \cdot v = (0 + 0) \cdot v - 0 \cdot v = (0 \cdot v + 0 \cdot v) - 0 \cdot v$$

$$= 0 \cdot v + (0 \cdot v - 0 \cdot v) = 0 \cdot v + \vec{0} = 0 \cdot v.$$

② Exercise

③ Observe that

$$v + (-1)v = 1\cdot v + (-1)\cdot v = (1-1)v = 0\cdot v = \vec{0}.$$

Since additive inverses are unique, it follows that $(-1)\cdot v = -v$. $\blacksquare$