2.2 Subsets and Spanning Sets

Def: A subcollection W of a vector space V is a subspace of V if W is also a vector space under the addition and scalar multiplication of V.

Ex: Let W be the subcollection of $\mathbb{R}^3$ consisting of all column vectors of the form

$$\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}.$$

Then addition and scalar multiplication are defined by

$$\begin{bmatrix} x_1 \\ y_1 \\ 0 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ 0 \end{bmatrix}$$

$$\lambda \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \\ 0 \end{bmatrix}.$$

Clearly W is closed under addition and closed under scalar multiplication. Furthermore, axioms 1 - 8 for a vector space are satisfied by W (exercise). Hence W is a subspace of $\mathbb{R}^3$. ∎

Ex: $GL_n(\mathbb{R})$ is a subcollection of the vector space $M_{n\times n}(\mathbb{R})$, but $GL_n(\mathbb{R})$ is not closed under addition: for $A \in GL_n(\mathbb{R})$, $A - A = 0_{n\times n} \notin GL_n(\mathbb{R})$. Hence $GL_n(\mathbb{R})$ is not a subspace of $M_{n\times n}(\mathbb{R})$. ∎

Thm: (One-Step Subspace Test) Let V be a vector space and let W be a subcollection of V. Then W is a subspace of V if and only if $\lambda_1 w_1 + \lambda_2 w_2 \in W$ for all scalars $\lambda_1, \lambda_2 \in \mathbb{R}$ and all vectors $w_1, w_2 \in W$.

Idea: To check that a subcollection W of a vector space V is a subspace, we need only check that $\lambda_1 w_1 + \lambda_2 w_2 \in W$ whenever $\lambda_1, \lambda_2 \in \mathbb{R}$ and $w_1, w_2 \in W$.

Ex: Let V be the subcollection of $\mathbb{R}^2$ consisting of all vectors of the form $\begin{bmatrix} x \\ 1 \end{bmatrix}$. Determine if V is a subspace of $\mathbb{R}^2$.

Solution: Note that

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \notin V.$$

So, V fails the one-step subspace test. Hence V is not a subspace of $\mathbb{R}^3$. ∎

Ex: Let V be the subcollection of $\mathbb{R}^3$ consisting of all vectors of the form $\begin{bmatrix} x \\ y \\ x-2y \end{bmatrix}$. Determine if V is a subspace of $\mathbb{R}^3$.

Solution: Use the one-step subspace test:

$$\lambda_1 \begin{bmatrix} x_1 \\ y_1 \\ x_1-2y_1 \end{bmatrix} + \lambda_2 \begin{bmatrix} x_2 \\ y_2 \\ x_2-2y_2 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 x_1 \\ \lambda_1 y_1 \\ \lambda_1(x_1-2y_1) \end{bmatrix} + \begin{bmatrix} \lambda_2 x_2 \\ \lambda_2 y_2 \\ \lambda_2(x_2-2y_2) \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 x_1 + \lambda_2 x_2 \\ \lambda_1 y_1 + \lambda_2 y_2 \\ \lambda_1(x_1-2y_1) + \lambda_2(x_2-2y_2) \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 x_1 + \lambda_2 x_2 \\ \lambda_1 y_1 + \lambda_2 y_2 \\ (\lambda_1 x_1 + \lambda_2 x_2) - 2(\lambda_1 y_1 + \lambda_2 y_2) \end{bmatrix} \in V$$

Hence V is a subspace of $\mathbb{R}^3$.

Thm: Let $V \in M_{m \times n}(\mathbb{R})$ and let V be the subcollection of $\mathbb{R}^n$ consisting of all solutions X to the homogeneous system $AX = 0$. Then V is a subspace of $\mathbb{R}^n$.

pf: Let $\lambda_1, \lambda_2 \in \mathbb{R}$ and let $v_1, v_2 \in V$. Then

$$
\begin{aligned}
A(\lambda_1 v_1 + \lambda_2 v_2) &= A(\lambda_1 v_1) + A(\lambda_2 v_2) \\
&= \lambda_1 A v_1 + \lambda_2 A v_2 \\
&= \lambda_1 0 + \lambda_2 0 \\
&= 0 + 0 \\
&= 0
\end{aligned}
$$

so that $\lambda_1 v_1 + \lambda_2 v_2 \in V$. Hence V is a subspace. ■

Ex: Let $\mathfrak{F}(a,b)$ ~~the~~ be the collection of all (not necessarily cts) functions defined on the open interval (a,b). Then $\mathfrak{F}(a,b)$ is a vector space.

Recall that $\mathcal{C}^k(a,b)$ is the vector space of all k-times differentiable functions defined on (a,b). Then $\mathcal{C}^k(a,b)$ is a subspace of $\mathfrak{F}(a,b)$. Furthermore, if $k < l$, then $\mathcal{C}^l(a,b)$ is a subspace of $\mathcal{C}^k(a,b)$.

Note that $\mathcal{C}^\infty(a,b)$ is the subspace of all infinitely differentiable functions on (a,b).

Ex: Let $\mathcal{P}$ be the collection of all polynomials. Elements of $\mathcal{P}$ are of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

where each $a_k \in \mathbb{R}$. Then $\mathcal{P}$ is a subspace of $C^{\infty}(-\infty,\infty)$.

Ex: Recall that the <u>degree</u> of a polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

is $\deg(p(x)) = n$. Let $\mathcal{P}_n$ be the subcollection of $\mathcal{P}$ consisting of all polynomials of degree $\leq n$. Show that $\mathcal{P}_n$ is a subspace of $\mathcal{P}$.

<u>Solution</u>: Use the one-step subspace test:

$$\lambda_1 \sum_{k=0}^{n} a_k x^k + \lambda_2 \sum_{k=0}^{n} b_k x^k$$

$$= \sum_{k=0}^{n} (\lambda_1 a_k) x^k + \sum_{k=0}^{n} (\lambda_2 b_k) x^k$$

$$= \sum_{k=0}^{n} \left((\lambda_1 a_k) x^k + (\lambda_2 b_k) x^k \right)$$

$$= \sum_{k=0}^{n} (\lambda_1 a_k + \lambda_2 b_k) x^k \in \mathcal{P}_n .$$

Hence $\mathcal{P}_n$ is a subspace of $\mathcal{P}$. $\blacksquare$

<u>Def</u>: Given a vector space V, let $v_1, v_2, \dots, v_n \in V$. A <u>linear combination</u> of $v_1, v_2, \dots, v_n$ is an expression of the form

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n$$

where $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}$.

<u>Thm</u>: Given a vector space V, let $v_1, v_2, \dots, v_n \in V$ and let W be the subcollection of V consisting of all linear combinations of $v_1, v_2, \dots, v_n$. Then W is a subspace of V.

<u>pf</u>: Use the one-step ~~subgroup~~ subspace test:

$$\lambda_1 \sum_{k=1}^{n} a_k v_k + \lambda_2 \sum_{k=1}^{n} b_k v_k$$

$$= \sum_{k=1}^{n} (\lambda_1 a_k) v_k + \sum_{k=1}^{n} (\lambda_2 b_k) v_k$$

$$= \sum_{k=1}^{n} \left((\lambda_1 a_k) v_k + (\lambda_2 b_k) v_k \right)$$

$$= \sum_{k=1}^{n} (\lambda_1 a_k + \lambda_2 b_k) v_k \in W.$$

Hence W is a subspace of V. ∎

Def: Given a vector space V, let $v_1, v_2, \ldots, v_n \in V$. Then the subspace of V consisting of all linear combinations of $v_1, v_2, \ldots, v_n$ is the subspace of V spanned by $v_1, v_2, \ldots, v_n$ and is denoted by $\operatorname{Span}\{v_1, v_2, \ldots, v_n\}$.

Ex: Is

$$\begin{bmatrix} 2 \\ -5 \\ 1 \\ 10 \end{bmatrix} \text{ in } \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\} ?$$

Solution: By the definition of Span, we wish to find scalars $\lambda_1, \lambda_2, \lambda_3$ such that

$$\lambda_1 \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix} + \lambda_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 1 \\ 10 \end{bmatrix}.$$

This gives the system

$$\begin{aligned} \lambda_1 + \lambda_2 - \lambda_3 &= 2 \\ -\lambda_1 - 2\lambda_2 &= -5 \\ 2\lambda_1 - \lambda_2 + \lambda_3 &= 1 \\ 3\lambda_1 + 2\lambda_2 + 3\lambda_3 &= 10. \end{aligned}$$

Now, since

$$\operatorname{rref}\left[\begin{array}{ccc:c} 1 & 1 & -1 & 2 \\ -1 & -2 & 0 & -5 \\ 2 & -1 & 1 & 1 \\ 3 & 2 & 3 & 10 \end{array}\right] = \left[\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

we have

$$\begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 1 \\ 10 \end{bmatrix}.$$

So, $\begin{bmatrix} 2 \\ -5 \\ 1 \\ 10 \end{bmatrix}$ is in the Span. ∎

<u>Ex</u>: Is $2x^2+x+1$ in $\operatorname{Span}\{x^2+x,\ x^2-1,\ x+1\}$?

<u>Solution</u>: ~~Eleme~~ We want scalars λ_1, λ_2, and λ_3 such that

$$\lambda_1(x^2+x) + \lambda_2(x^2-1) + \lambda_3(x+1) = 2x^2+x+1.$$

This gives the system

$$\begin{aligned} \lambda_1 + \lambda_2 &= 2 \\ \lambda_1 + \lambda_3 &= 1 \\ -\lambda_2 + \lambda_3 &= 1. \end{aligned}$$

Since

$$\operatorname{rref}\left[\begin{array}{ccc:c} 1 & 1 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 \end{array}\right] = \left[\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

is not consistent, the system has no solution. Hence $2x^2+x+1$ is not in $\operatorname{Span}\{x^2+x,\ x^2-1,\ x+1\}$. ∎

Ex: Is $\text{Span}\left\{\begin{bmatrix}1\\-2\end{bmatrix}, \begin{bmatrix}2\\-4\end{bmatrix}\right\} = \mathbb{R}^2$?

Solution: We want scalars λ_1, λ_2 such that

$$\lambda_1\begin{bmatrix}1\\-2\end{bmatrix} + \lambda_2\begin{bmatrix}2\\-4\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix}.$$

This gives the system

$$\begin{aligned}\lambda_1 + 2\lambda_2 &= a\\ -2\lambda_1 - 4\lambda_2 &= b.\end{aligned}$$

Since

$$\left[\begin{array}{cc|c}1 & 2 & a\\ -2 & -4 & b\end{array}\right] \longrightarrow \left[\begin{array}{cc|c}1 & 2 & a\\ 0 & 0 & b+2a\end{array}\right],$$

we see that the system is inconsistent whenever $b+2a \neq 0$. So, the system does not have a solution for all a and b. Hence

$\text{Span}\left\{\begin{bmatrix}1\\-2\end{bmatrix}, \begin{bmatrix}2\\-4\end{bmatrix}\right\} \neq \mathbb{R}^2$. $\blacksquare$

Ex: Is $\text{Span}\{x^2+x-3, x-5, 3\} = P_2$?

Solution: We want scalars $\lambda_1, \lambda_2, \lambda_3$ such that

$$\lambda_1(x^2+x-3) + \lambda_2(x-5) + \lambda_3 \cdot 3 = ax^2+bx+c.$$

This gives the system

$$\begin{aligned}\lambda_1 &= a\\ \lambda_1 + \lambda_2 &= b\\ -3\lambda_1 - 5\lambda_2 + 3\lambda_3 &= c,\end{aligned}$$

which has a solution (Exercise). $\blacksquare$