NOTES ON KEVIN COSTELLO'S "THE A_∞ OPERAD AND THE MODULI SPACE OF CURVES"

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Definition 1. *A* **graph** *is a tuple* $\gamma = \big(V(\gamma), H(\gamma), \pi, \sigma\big)$ *consisting of*

- *a finite set* $V(\gamma)$ *whose elements are called the* **vertices** *of* γ
- *a finite set* $H(\gamma)$ *whose elements are called the* **half edges** *of* γ
- *a map* $\pi : H(\gamma) \to V(\gamma)$
- *an involution* σ *on* $H(\gamma)$.

Definition 2. *Let* $\gamma = \big(V(\gamma), H(\gamma), \pi, \sigma\big)$ *be a graph.*

- *The* **edges** *of* γ *are the elements of* $E(\gamma) = \operatorname{orb}(\sigma)$.
- *The* **tails** *of* γ *are the elements of* $T(\gamma) = \operatorname{fix}(\sigma)$.
- *The* **connected components** *of* γ *are the elements of* $C(\gamma) = V(\gamma)/\sim$ *where* $\sim$ *is the smallest equivalence relation satisfying* $v \sim w$ *if there exists an* $h \in H(\gamma)$ *such that* $\pi(h) = v$ *and* $\pi\big(\sigma(h)\big) = w$.
- *The* **tail map** *of* γ *is the map* $\iota : T(\gamma) \to C(\gamma)$ *given by* $\iota(h) = \big[\pi(h)\big]$.
- *The* **path relation** *on* γ *is the relation* $\sim_p$ *on* $V(\gamma)$ *given by* $v \sim_p w$ *if and only if there exists a map* $f : [2n] \to H(\gamma)$ *such that*
  - $\Big\{\big\{f(k), f(k+1)\big\} : 0 \le k \le 2n-2\Big\} \subseteq E(\gamma)$
  - $f(0) = v$
  - $f(2n-1) = w$.

Definition 3. *A* **forest** *is a graph* $\gamma = \big(V(\gamma), H(\gamma), \pi, \sigma\big)$ *such that the fibres of* π *are of cardinality at least three and whose path relation is irreflexive.*

Definition 4. *A* **rooted forest** *is a pair* (γ, s) *consisting of a forest* γ *and a section* s *of the tail map of* γ.

Definition 5. *Let* **Graphs** *be the category where*

- $\operatorname{ob}(\mathbf{Graphs}) = \operatorname{ar}(\mathbf{set} \downarrow \mathbf{set})/\approx$
- $\operatorname{ar}(\mathbf{Graphs})$ *is the collection of all graphs*
- $\operatorname{dom}(\gamma) = [\pi_\gamma]$
- $\operatorname{cod}(\gamma) = [\iota_\gamma]$
- $\mathbf{1}_{[f]}$ *is a graph* (V, H, π, σ) *satisfying*
  - $V \approx \operatorname{cod}(f)$
  - $H \approx \operatorname{dom}(f)$
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