LOCALLY SIERPINSKI JULIA SETS OF WEIERSTRASS ELLIPTIC $\wp$ FUNCTIONS

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Abstract. We define a locally Sierpinski Julia set to be a Julia set of an elliptic function which is a Sierpinski curve in each fundamental domain for the lattice. We give sufficient conditions for which a Weierstrass elliptic $\wp$ function is quadratic-like, and we then use these results to prove the existence of locally Sierpinski Julia sets for certain elliptic functions. We show this results in naturally occurring Sierpinski curves in the plane, sphere, and torus as well.

1. Introduction

A Sierpinski curve, also called a Sierpinski carpet, is a planar set that contains a homeomorphic copy of any compact, connected one topological dimensional planar set. It was introduced by Sierpinski in 1916 and was described by G. T.Whyburn [25] to be the following set:

> The curve is obtained very simply as the residual set remaining when one begins with a square and applies the operation of dividing it into nine equal squares and omitting the interior of the center one, then repeats the operation on each of the surviving 8 squares, then repeats again on the surviving 64 squares, and so on indefinitely.

A characterization of a Sierpinski curve, defined to be a set homeomorphic to the set described above, was given in [25] where it was shown that a set satisfying the following definition is homeomorphic to the classical Sierpinski carpet and is therefore a universal planar set.

Definition 1.1. An $\mathcal{S}$-curve, is a subset S of the plane such that

(1) S is compact;
(2) S is connected; (the first two properties are that S should be a planar continuum.)
(3) S is nowhere dense; (an $\mathcal{S}$-curve has topological dimension one);
(4) S is locally connected;

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(5) $S = \mathbb{C} \setminus \cup_\alpha U_\alpha$, where each U_α is a simply connected open set and ∂U_α's are pairwise disjoint simple closed curves.

Whenever S is a subset of a 2-dimensional manifold M (not necessarily the plane) satisfying Properties (1)-(5), then we call S a *Sierpinski curve.*

Remark 1.1. The following from [25] is used in later examples in Section 5. Let γ be any simple closed curve lying in an $\mathcal{S}$-curve S, and let I denote the interior of γ. If no complementary domain boundary ∂U lying in $I \cup \gamma$ intersects γ, then $T = S \cap (\gamma \cup I)$ is again an $\mathcal{S}$-curve.

The properties listed in Definition 1.1 are immediately reminiscent of properties of Julia sets arising in complex dynamics of rational maps, so this is a natural place in which to search for such curves, and indeed Milnor and Tan Lei in 1993 were the first to give an example [22]. Soon thereafter many parametrized families of examples were produced by Devaney, the second author, and others in [5],[9], and [10], and other publications. This work is nicely summarized in [8].

Recent studies on the Weierstrass elliptic $\wp$ function, [14], [15], [16],and [19] suggest that Sierpinski curves occur as Julia sets in this setting as well; unlike the rational setting, instead of getting a single Sierpinski carpet for a Julia set in this setting we obtain a "Sierpinski carpet tile" that then is used to tile the entire plane. We refer to this phenomenon as *wall-to-wall Sierpinski carpeting* and say the Julia set is *locally Sierpinski.* The Julia set of an elliptic function is not compact in the plane due to the double periodicity with respect to a lattice; so Property 1 in Definition 1.1 is never satisfied. Therefore we extend the definition to make precise the notion of wall-to-wall Sierpinski carpeting; (all of the terms used in Definition 1.2 are defined in Section 2 of this paper).

Definition 1.2. A *locally Sierpinski Julia set* of an elliptic function f is a Julia set $J(f)$ of a meromorphic function f with the property that there exists a period parallelogram Q for the period lattice Λ such that $J(f) \cap Q$ is a Sierpinski curve.

However, on $\mathbb{C}_\infty = \mathbb{C} \cup \infty$, the Riemann sphere, we have a compact Julia set and we will show that for some elliptic functions we have Sierpinski curve Julia sets on the sphere. Furthermore, we can view the Julia sets on the compact torus $\mathbb{C}/\Gamma$ where Γ is the lattice of periods of the functions; so we obtain in a natural way Sierpinski curves on the torus. The purpose of this paper is to prove the existence of locally Sierpinski Julia sets for certain lattices and for many Weierstrass elliptic $\wp$ functions that are not pairwise conformally conjugate. In order to

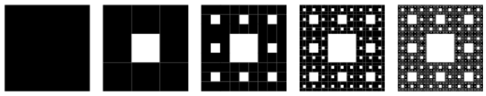

FIGURE 1. The “original” Sierpinski curve

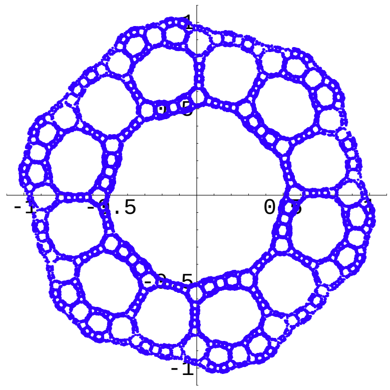

FIGURE 2. A Sierpinski curve Julia set for a rational map

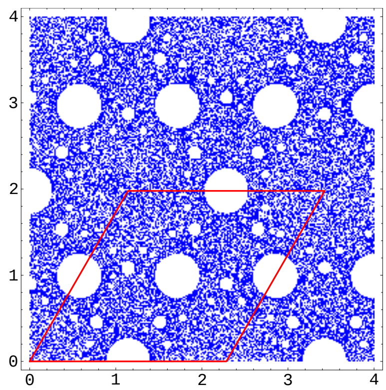

FIGURE 3. A locally Sierpinski Julia set for an elliptic map with period lattice shown

do this, and of independent interest, we give sufficient conditions (on the lattice) for which an elliptic $\wp$ function is quadratic-like.

In Figure 2 we show a Sierpinski curve Julia set for the rational map of the form $R(z) = z^3 + \lambda/z^3$ studied by the second author in [10], while in Figure 3 we show a Julia set comprised of wall-to-wall Sierpinski carpeting. In Figure 4 we show one Sierpinski curve carpet tile. In each of the figures, the Sierpinski Julia set is represented by the blue points.

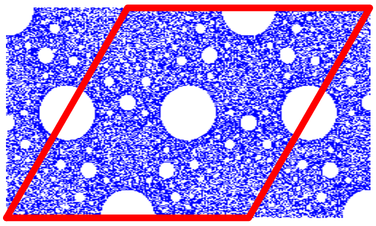

FIGURE 4. A single Sierpinski carpet tile

The paper is organized as follows. In Section 2 we outline the basic dynamics of elliptic functions. Section 3 summarizes some results on hyperbolic elliptic functions and connected Julia sets. The main results of the paper are given in Sections 4 and 5. We first give sufficient conditions under which a Weierstrass elliptic $\wp$ function is quadratic like, and we then use these results in Section 5 to prove the existence of locally Sierpinski Julia sets for elliptic functions.

The first author thanks Bob Devaney for his hospitality and fruitful discussions during a visit to Boston University, and the second author thanks him for his supervision of his Ph.D. thesis on topics related to the subject of this paper.

2. Preliminaries on the dynamics of Weierstrass elliptic $\wp$ function

Let $\lambda_1, \lambda_2 \in \mathbb{C}\backslash\{0\}$ such that $\lambda_2/\lambda_1 \notin \mathbb{R}$, and define $\Lambda = [\lambda_1, \lambda_2]\colon = \{m\lambda_1 + n\lambda_2\colon m, n \in \mathbb{Z}\}$; the generators are not unique. We view Λ as a group acting on $\mathbb{C}$ by translation, each $\omega \in \Lambda$ inducing the transformation of $\mathbb{C}$:

$$T_\omega : z \mapsto z + \omega.$$

Definition 2.1. A closed, connected subset Q of $\mathbb{C}$ is defined to be a *fundamental region* for Λ if

(1) for each $z \in \mathbb{C}$, Q contains at least one point in the same Λ-orbit as z;
(2) no two points in the interior of Q are in the same Λ-orbit.

If Q is any fundamental region for Λ, then for any $s \in \mathbb{C}$, the set

$$Q + s = \{z + s : z \in Q\}$$

is also a fundamental region. If Q is a parallelogram we call it a *period parallelogram* for Λ.

The "appearance" of a lattice $\Lambda = [\lambda_1, \lambda_2]$ is determined by the ratio $\tau = \lambda_2/\lambda_1$. (We generally choose the generators so that $Im\,(\tau) > 0$.) If $\Lambda = [\lambda_1, \lambda_2]$, and $k \neq 0$ is any complex number, then $k\Lambda$ is the lattice defined by taking $k\lambda$ for each $\lambda \in \Lambda$; $k\Lambda$ is said to be *similar* to Λ. Similarity is an equivalence relation between lattices, and an equivalence class of lattices is called a *shape*.

Definition 2.2.
(1) $\Lambda = [\lambda_1, \lambda_2]$ is *real rectangular* if there exist generators such that λ_1 is real and λ_2 is purely imaginary. Any lattice similar to a real rectangular lattice is *rectangular*.
(2) $\Lambda = [\lambda_1, \lambda_2]$ is *real rhombic* if there exist generators such that $\lambda_2 = \overline{\lambda_1}$. Any similar lattice is *rhombic*.
(3) A lattice Λ is *square* if $i\Lambda = \Lambda$. (Equivalently, Λ is square if it is similar to a lattice generated by $[\lambda, \lambda\, i]$, for some $\lambda > 0$.)
(4) A lattice Λ is *triangular* if $\Lambda = e^{2\pi i/3}\Lambda$ in which case a period parallelogram can be made from two equilateral triangles.

In each of cases (1) − (3) the period parallelogram with vertices $0, \lambda_1, \lambda_2$, and $\lambda_3 := \lambda_1 + \lambda_2$ can be chosen to be a rectangle, rhombus, or square respectively.

Definition 2.3. An *elliptic function* is a meromorphic function in $\mathbb{C}$ which is periodic with respect to a lattice Λ.

For any $z \in \mathbb{C}$ and any lattice Λ, the *Weierstrass elliptic function* is defined by

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{w \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right).$$

Replacing every z by $-z$ in the definition we see that $\wp_\Lambda$ is an even function. The map $\wp_\Lambda$ is meromorphic, periodic with respect to Λ, and has order 2.

The derivative of the Weierstrass elliptic function is also an elliptic function which is periodic with respect to Λ defined by

$$\wp'_\Lambda(z) = -2 \sum_{w \in \Lambda} \frac{1}{(z-w)^3}.$$

The Weierstrass elliptic function and its derivative are related by the differential equation

$$\wp'_\Lambda(z)^2 = 4\wp_\Lambda(z)^3 - g_2\wp_\Lambda(z) - g_3, \tag{1}$$

where $g_2(\Lambda) = 60 \sum_{w \in \Lambda \setminus \{0\}} w^{-4}$ and $g_3(\Lambda) = 140 \sum_{w \in \Lambda \setminus \{0\}} w^{-6}$.

The numbers $g_2(\Lambda)$ and $g_3(\Lambda)$ are invariants of the lattice Λ in the following sense: if $g_2(\Lambda) = g_2(\Lambda')$ and $g_3(\Lambda) = g_3(\Lambda')$, then $\Lambda = \Lambda'$.

Furthermore given any g_2 and g_3 such that $g_2^3 - 27g_3^2 \neq 0$ there exists a lattice Λ having $g_2 = g_2(\Lambda)$ and $g_3 = g_3(\Lambda)$ as its invariants [12].

Theorem 2.1. [12] *For $\Lambda_\tau = [1, \tau]$, the functions $g_i(\tau) = g_i(\Lambda_\tau), i = 2, 3$, are analytic functions of τ in the open upper half plane $Im(\tau) > 0$.*

We have the following homogeneity in the invariants g_2 and g_3 [15].

Lemma 2.2. *For lattices Λ and Λ', $\Lambda' = k\Lambda \Leftrightarrow$*

$$g_2(\Lambda') = k^{-4} g_2(\Lambda) \quad \text{and} \quad g_3(\Lambda') = k^{-6} g_3(\Lambda).$$

Theorem 2.3. [17] *The following are equivalent:*

1. $\wp_\Lambda(\overline{z}) = \overline{\wp_\Lambda(z)}$;
2. Λ *is a real lattice;*
3. $g_2, g_3 \in \mathbb{R}$.

For any lattice Λ, the Weierstrass elliptic function and its derivative satisfy the following properties: for $k \in \mathbb{C} \backslash \{0\}$,

$$\wp_{k\Lambda}(ku) = \frac{1}{k^2} \wp_\Lambda(u), \text{ (homogeneity of } \wp_\Lambda), \tag{2}$$

$$\wp'_{k\Lambda}(ku) = \frac{1}{k^3} \wp'_\Lambda(u), \text{ (homogeneity of } \wp'_\Lambda).$$

The critical points and values of the Weierstrass elliptic function on an arbitrary lattice $\Lambda = [\lambda_1, \lambda_2]$ are as follows. For $j = 1, 2$, we have $\wp_\Lambda(\lambda_j - z) = \wp_\Lambda(z)$ for all z. Taking derivatives of both sides we obtain $-\wp'_\Lambda(\lambda_j - z) = \wp'_\Lambda(z)$, so at $z = \lambda_1/2, \lambda_2/2$, or $\lambda_3/2$, we see that $\wp'_\Lambda(z) = 0$. We use the notation

$$e_1 = \wp_\Lambda(\frac{\lambda_1}{2}), \quad e_2 = \wp_\Lambda(\frac{\lambda_2}{2}), \quad e_3 = \wp_\Lambda(\frac{\lambda_3}{2})$$

to denote the critical values. Since e_1, e_2, e_3 are the (distinct) zeros of Equation 1, we also write

$$\wp'_\Lambda(z)^2 = 4(\wp_\Lambda(z) - e_1)(\wp_\Lambda(z) - e_2)(\wp_\Lambda(z) - e_3). \tag{3}$$

Equating like terms in Equations 1 and 3, we obtain

$$e_1 + e_2 + e_3 = 0, \quad e_1e_3 + e_2e_3 + e_1e_2 = \frac{-g_2}{4}, \quad e_1e_2e_3 = \frac{g_3}{4}. \tag{4}$$

In general, similar lattices do not result in conformally conjugate elliptic functions [15]. We can start with a fixed shape lattice, say $\Lambda = [1, \tau]$ and produce many types of dynamics as was shown in [14]-[16]. Within a given simlarity class (shape), Equation (2), is used to produce infinitely many lattices with fixed critical points (i.e. with real superattracting fixed points). These examples will be shown in many

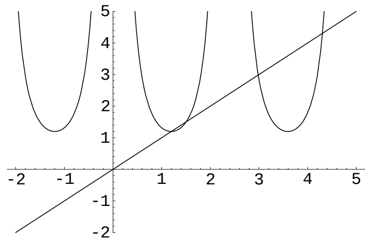

FIGURE 5. Graph of $\wp_\Lambda|_\mathbb{R}$ with a superattracting fixed point

cases to yield locally Sierpinski Julia sets. The following proposition generalizes a result from [15] for real lattices.

Proposition 2.1. Let $\Lambda = [1, \tau]$ be a lattice such that the critical value $\wp_\Lambda(1/2) = \epsilon \neq 0$. If m is any odd integer and $k = \sqrt[3]{2\epsilon/m}$ (taking any root) then the lattice $\Gamma = k\Lambda$ has a superattracting fixed point at $mk/2$.

Proof. Equation (2) for $\wp'_{k\Lambda}$ implies that $k/2$ is a critical point for $\wp_\Gamma$. Since m is odd, periodicity implies that $\wp_\Gamma(mk/2) = \wp_\Gamma(k/2)$. Further, the homogeneity property implies that

$$\wp_\Gamma(\frac{mk}{2}) = \wp_\Gamma(\frac{k}{2}) = \wp_{k\Lambda}(\frac{k}{2}) = \frac{1}{k^2}\wp_\Lambda(\frac{1}{2}) = \frac{\epsilon}{k^2} = \frac{mk}{2}.$$

□

Corollary 2.4. Every similarity class contains a lattice Λ for which $\wp_\Lambda$ has a superattracting fixed point.

Proof. We apply Proposition 2.1, and we need only to check that $\wp_\Lambda(\frac{1}{2}) = \epsilon \neq 0$ for each (nonreal) choice of τ. A critical value $e_j = 0$ if and only if $g_3 = 0$, by Equation 4. This holds if and only if the lattice is square, in which case $\tau = i$. However in this case we know that $e_3 = 0$, and $e_1 = \wp_\Lambda(1/2) \neq 0$.

□

Figure 5 shows the graph of $\wp_\Lambda$ on $\mathbb{R}$ for a real square lattice which has a superattracting fixed point.

2.1. **Fatou and Julia sets for elliptic functions.** We review the basic dynamical definitions and properties for meromorphic functions which appear in [1], [4], [7] and [8]. Let $f\colon \mathbb{C} \to \mathbb{C}_\infty$ be a meromorphic function. The *Fatou set* $F(f)$ is the set of points $z \in \mathbb{C}_\infty$ such that $\{f^n\colon n \in \mathbb{N}\}$ is defined and normal in some neighborhood of z. The *Julia set* is the complement of the Fatou set on the sphere, $J(f) =$

$\mathbb{C}_\infty \backslash F(f)$. Notice that $\mathbb{C}_\infty \backslash \overline{\bigcup_{n\geq 0} f^{-n}(\infty)}$ is the largest open set where all iterates are defined. Since $f(\mathbb{C}_\infty \backslash \overline{\bigcup_{n\geq 0} f^{-n}(\infty)}) \subset \mathbb{C}_\infty \backslash \overline{\bigcup_{n\geq 0} f^{-n}(\infty)}$, Montel's theorem implies that

$$J(f) = \overline{\bigcup_{n\geq 0} f^{-n}(\infty)}.$$

Let $Crit(f)$ denote the set of critical points of f, *i.e.*,

$$Crit(f) = \{z \colon f'(z) = 0\}.$$

If z_0 is a critical point then $f(z_0)$ is a *critical value*. For each lattice, $\wp_\Lambda$ has three critical values and no asymptotic values. The *singular set* $Sing(f)$ of f is the set of critical and finite asymptotic values of f and their limit points. A function is called *Class S* if f has only finitely many critical and asymptotic values; for each lattice Λ, every elliptic function with period lattice Λ is of Class S. The *postcritical set* of $\wp_\Lambda$ is:

$$P(\wp_\Lambda) = \overline{\bigcup_{n\geq 0} \wp_\Lambda^n(e_1 \cup e_2 \cup e_3)}.$$

For a meromorphic function f, a point z_0 is *periodic* of period p if there exists a $p \geq 1$ such that $f^p(z_0) = z_0$. We also call the set $\{z_0, f(z_0), \dots, f^{p-1}(z_0)\}$ a *p-cycle.* The *multiplier* of a point z_0 of period p is the derivative $(f^p)'(z_0)$. A periodic point z_0 is called *attracting, repelling,* or *neutral* if $|(f^p)'(z_0)|$ is less than, greater than, or equal to 1 respectively. If $|(f^p)'(z_0)| = 0$ then z_0 is called a *superattracting* periodic point. As in the case of rational maps, the Julia set is the closure of the repelling periodic points [1].

Suppose U is a connected component of the Fatou set. We say that U is *preperiodic* if there exists $n > m \geq 0$ such that $f^n(U) = f^m(U)$, and the minimum of $n - m = p$ for all such n, m is the *period* of the cycle.

Since every elliptic function is of Class S the basic dynamics are similar to those of rational maps with the exception of the poles. The first result holds for all Class S functions as was shown in [4] (Theorem 12) and [23].

Theorem 2.5. *For any lattice Λ, the Fatou set of an elliptic function f_Λ with period lattice Λ has no wandering domains and no Baker domains.*

In particular, Sullivan's No Wandering Domains Theorem holds in this setting so all Fatou components of f_Λ are preperiodic. Because there are only finitely many critical values, we have a bound on the number of attracting periodic points that can occur.

The next result was proved in [15]; it is only known for the Weierstrass elliptic function.

Theorem 2.6. *For any lattice Λ, $\wp_\Lambda$ has no cycle of Herman rings.*

We summarize this discussion with the following result.

Theorem 2.7. *For any lattice Λ, at most three different types of forward invariant Fatou cycles can occur for $\wp_\Lambda$, and each periodic Fatou component contains one of these:*

1. *a linearizing neighborhood of an attracting periodic point;*
2. *a Böttcher neighborhood of a superattracting periodic point;*
3. *an attracting Leau petal for a periodic parabolic point. The periodic point is in $J(\wp_\Lambda)$;*
4. *a periodic Siegel disk containing an irrationally neutral periodic point.*

2.1.1. *Symmetries of the Julia and Fatou sets.*

Lemma 2.8. *If Λ is any lattice and f_Λ is an elliptic function with period lattice Λ, then*

1. $J(f_\Lambda) + \Lambda = J(f_\Lambda)$, *and*
2. $F(f_\Lambda) + \Lambda = F(f_\Lambda)$.

The algebraic and analytic symmetry of the Weierstrass elliptic function manifests itself in a large amount of symmetry in each Julia set arising from an elliptic function. The proof of Theorem 2.9 is given in [14].

Theorem 2.9. *If $\Lambda = [\lambda_1, \lambda_2]$ is any lattice then*

1. $J(\wp_\Lambda) + \Lambda = J(\wp_\Lambda)$ *and* $F(\wp_\Lambda) + \Lambda = F(\wp_\Lambda)$.
2. $(-1)J(\wp_\Lambda) = J(\wp_\Lambda)$ *and* $(-1)F(\wp_\Lambda) = F(\wp_\Lambda)$.
3. $\overline{J(\wp_\Lambda)} = J(\wp_{\overline{\Lambda}})$ *and* $\overline{F(\wp_\Lambda)} = F(\wp_{\overline{\Lambda}})$.
4. *If Λ is square, then* $e^{\pi i/2}J(\wp_\Lambda) = J(\wp_\Lambda)$ *and* $e^{\pi i/2}F(\wp_\Lambda) = F(\wp_\Lambda)$.
5. *If Λ is triangular, then* $e^{2\pi i/3}J(\wp_\Lambda) = J(\wp_\Lambda)$ *and* $e^{2\pi i/3}F(\wp_\Lambda) = F(\wp_\Lambda)$. *Moreover,* $e^{2\pi i/3}\wp_\Lambda(z) = \wp_\Lambda(e^{2\pi i/3}z)$ *for all* $z \in \mathbb{C} \setminus \Lambda$.

In addition to a basic Julia set pattern repeating on each fundamental period, we also see symmetry within the period parallelogram.

Proposition 2.2. For the lattice $\Lambda = [\lambda_1, \lambda_2]$, $J(\wp_\Lambda)$ and $F(\wp_\Lambda)$ are symmetric with respect to any critical point $\lambda_1/2 + \Lambda$, $\lambda_2/2 + \Lambda$, and $(\lambda_1+\lambda_2)/2+\Lambda$. That is, if c is any critical point of $\wp_\Lambda$, then $c+z \in J(\wp_\Lambda)$ if and only if $c - z \in J(\wp_\Lambda)$.

In particular, if F_o is any component of $F(\wp_\Lambda)$ that contains a critical point c, then F_o is symmetric with respect to c.

3. Connected Julia sets and hyperbolic Weierstrass $\wp_\Lambda$ functions

We give sufficient conditions under which $J(\wp_\Lambda)$ is connected if $\wp_\Lambda$ is the Weierstrass elliptic $\wp$ function with period lattice Λ. The proof of the next result appears in [16]. It uses the fact that although there are infinitely many critical points for $\wp_\Lambda$, there are exactly three critical values and $\wp_\Lambda$ is locally two-to-one in each fundamental region.

Theorem 3.1. *If Λ is a lattice such that each critical value of $\wp_\Lambda$ that lies in the Fatou set is the only critical value in that component, then $J(\wp_\Lambda)$ is connected. In particular, if each Fatou component contains either 0 or 1 critical value, then $J(\wp_\Lambda)$ is connected.*

We now turn to the definition of a hyperbolic elliptic function; the concept of hyperbolicity for a meromorphic function is similar to that for a rational map, but the equivalent notions in the rational settings are not always the same for meromorphic functions (e.g., see [23]).

Definition 3.1. We say that an elliptic function is *hyperbolic* if $J(f)$ is disjoint from $P(f)$.

We adapt the following result from [23] (Theorem C) to our setting. We define the set

$$A_n(f) = \{z \in \mathbb{C} : f^n \text{ is not analytic at } z\}.$$

Since f is elliptic,

$$A_n(f) = \bigcup_{j=1}^{n} f^{-j}(\{\infty\}).$$

We therefore denote the set of prepoles for f by

$$\mathcal{A} = \bigcup_{n \geq 1} A_n.$$

We say that ω is an *order k prepole* if $\wp_\Lambda^k(\omega) = \infty$; so a pole is an order 1 prepole.

Theorem 3.2. [23] *If an elliptic function f is hyperbolic then there exist $K > 1$ and $c > 0$ such that*

$$|(f^n)'(z)| > cK^n \frac{|f^n(z)| + 1}{|z| + 1},$$

for each $z \in J(f) \setminus A_n(f), n \in \mathbb{N}$.

Remark 3.1. For a fixed lattice Λ, and elliptic function $f = f_\Lambda$, we have that $f(z+\lambda) = f(z)$ and $f'(z+\lambda) = f'(z)$ for every $\lambda \in \Lambda$. Therefore we can prove the following standard result about hyperbolic functions; the proof appears in [16].

Theorem 3.3. *An elliptic function f is hyperbolic if and only if there exist $K > 1$ and $C > 0$, $C = C(\Lambda)$, such that*

$$|(f^n)'(z)| > CK^n, \tag{5}$$

for each $z \in J(f) \setminus A_n(f), n \in \mathbb{N}$.

Corollary 3.4. If f is hyperbolic, then there exists an $r > 1$, an $s \in \mathbb{N}$, and a neighborhood U of $J(f)$, such that

$$|(f^s)'(z)| > r \quad \text{for all}\, z \in U \setminus A_s(f).$$

3.1. **Hyperbolic elliptic functions on triangular lattices.** We begin with some properties of $\wp_\Lambda$ for Λ a triangular period lattice.

Proposition 3.1. [15] and [16] Assume Λ is triangular.

(1) Then $g_2 = 0$; in this case e_1, e_2, e_3 all have the same modulus and are cube roots of $g_3/4$. Furthermore, e_i is real for some $i = 1, 2, 3$ if and only if g_3 is real, if and only if Λ is a real lattice.
(2) $g_3 > 0$ if and only if some $e_j > 0$, in which case there exists $\lambda > 0$ such that $\Lambda = [\lambda e^{\pi i/3}, \lambda e^{-\pi i/3}]$.
(3) The postcritical set $P(\wp_\Lambda)$ is contained in three forward invariant sets: one set $\alpha = \overline{\bigcup_{n\geq 0} \wp_\Lambda^n(e_1)}$, and the sets $e^{2\pi/3}\alpha$ and $e^{4\pi/3}\alpha$. (These sets are not necessarily disjoint.)
(4) $J(\wp_\Lambda)$ is connected.

Corollary 3.5. Assume Λ is triangular and there is an attracting periodic point. Then the following hold:

(1) If p_0 denotes the attracting periodic point, then $p_1 = e^{2\pi/3}p_0$ and $p_2 = e^{4\pi/3}p_0$ are attracting periodic points as well.
(2) $\wp_\Lambda$ is hyperbolic.

The main result of this section then is the following.

Theorem 3.6. *If Λ is a triangular lattice and $\wp_\Lambda$ has an attracting cycle, then $J(\wp_\Lambda)$ is hyperbolic and connected .*

Proof. The result follows from Proposition 3.1 and Corollary 3.5. □

In order to determine which triangular lattices give rise to Fatou sets such that the boundary of each simply connected component is a simple closed curve, we need to apply the theory of polynomial-like mappings (see Figures 3 and 4).

4. Quadratic-like elliptic functions

In this section we show that for many lattices the Weierstrass elliptic $\wp$ function acts locally like a quadratic polynomial in the sense introduced by Douady and Hubbard in [11]. (This was conjectured and illustrated by the first author and Koss in [15].) While the section is of independent interest, it is also useful for determining local connectivity of the Julia set of a Weierstrass elliptic function. For the discussion which follows we need the following results from [12] regarding period parallelograms for elliptic functions.

Proposition 4.1. If $\Lambda = [\lambda_1, \lambda_2]$ is any lattice and $u \in \mathbb{C}$, let $\Omega_u = \{u + s\lambda_1 + t\lambda_2 : 0 \leq s, t < 1\}$. Then $\wp_\Lambda : \Omega_u \to \mathbb{C}_\infty$ is onto and two-to-one except at the points that lie in the Λ-orbit of 0, $\lambda_1/2$, $\lambda_2/2$, and $(\lambda_1 + \lambda_2)/2$ (the critical points).

Corollary 4.1. For any lattice Λ, if u is a lattice point or half lattice point, then Ω_u is a fundamental period parallelogram containing in its interior no poles and one critical point, and $\wp_\Lambda : \Omega_u \to \mathbb{C}_\infty$ is analytic, onto, and two-to-one counting multiplicity.

Let $int\, \Omega_u$ denote the interior of Ω_u. Our goal is to export to our setting the following result from Mañé, Sad, and Sullivan [20]. A simple closed curve γ is a *quasicircle* if it is the image of a circle under a quasiconformal homeomorphism (of the sphere); quasicircles are locally connected.

Theorem 4.2. *If p is a quadratic polynomial with an attracting fixed point, then $J(p)$ is a quasicircle.*

Definition 4.1. Let U and V be simply connected bounded open subsets of $\mathbb{C}$ such that $\overline{U} \subset V$ is compact. A map $f : U \to V$ is a *polynomial-like map* if f is a d-fold covering map. A *quadratic-like map* has $d = 2$. A polynomial-like map of degree d has $d - 1$ critical points in U.

We denote a quadratic-like map by $(f : U, V)$. The *filled Julia set* K_f of $(f : U, V)$ is the set of points which do not escape U under f, i.e.,

$$K_f = \{z \in U | f^n(z) \in U \quad \text{for all} \quad n \geq 0\}.$$

Two polynomial-like maps f and g are *hybrid equivalent* if there is a quasiconformal conjugacy ϕ between f and g defined on a neighborhood of their filled Julia sets.

The following result was shown in [11] in the setting of rational maps, but since it is a local result, it applies equally well to elliptic functions.

Proposition 4.2. For a polynomial-like map f, $K(f)$ is connected if and only if it contains every critical point of f. If $K(f)$ is connected, and f is of degree d then f is hybrid equivalent to a polynomial of degree d.

We are now in a position to prove:

Theorem 4.3. *Consider a lattice Λ such that $\wp_\Lambda$ has the following properties:*

- *$\wp_\Lambda$ has an attracting fixed point q;*
- *the forward invariant Fatou component F_o containing q has exactly one critical value and one critical point in it;*
- *there is a period parallelogram Ω_u, with u a lattice or half lattice point, such that*
$$\overline{F_o} \subset int\,\Omega_u.$$
- *there are no other critical values of $\wp_\Lambda$ contained in $int\,\Omega_u$.*

We define $\mathcal{P} = \wp_\Lambda|_{\Omega_u}$. Then there exists an open set $\mathcal{O} \subset \Omega_u$ such that $(\mathcal{P} : \mathcal{O}, \mathcal{P}(\mathcal{O}))$ is a quadratic-like map and $\partial K(\mathcal{P})$ is a quasicircle.

Proof. Let $\Lambda = [\lambda_1, \lambda_2]$ be a lattice satisfying the hypotheses above. Our hypothesis on F_o implies that it is simply connected and contained in one fundamental period by [15]. Furthermore F_o contains the fixed point q of $\wp_\Lambda$. By hypothesis we can find a lattice or half lattice point u satisfying the hypothesis, and applying Corollary 4.1, there are poles of $\wp_\Lambda$ on the boundary of the parallelogram Ω_u and none lie in the interior; so $\wp_\Lambda$ is analytic and contains exactly one critical value on $int\,\Omega_u$. We define the sets $V = int\,\Omega_u$ and $\mathcal{O} = \mathcal{P}^{-1}(V)$; i.e.,$\mathcal{O}$ is the connected component of $\wp_\Lambda^{-1}V$ containing q. Clearly $\mathcal{P}(\mathcal{O}) = V$. Since $\mathcal{P}$ maps Ω_u two-to-one onto $\mathbb{C}_\infty$, and all of the poles of $\mathcal{P}$ are on the boundary $\mathcal{O}$, we have:

(1) $\mathcal{O}$ is homeomorphic to a disk,
(2) $\wp_\Lambda$ is degree 2 on $\mathcal{O}$, and
(3) $\wp_\Lambda(\mathcal{O})$ contains $\mathcal{O}$ and is homeomorphic to a disk.

The first two statements are clear; we prove the third. Since V contains exactly one critical value, then $\wp_\Lambda^{-1}(V)$ is a single simply connected region in each fundamental period, bounded by a simple closed curve and containing a critical point on the interior of each parallelogram of the form $\Omega_{(u+m\lambda_1+n\lambda_2)}, m, n \in \mathbb{Z}$; the set $\mathcal{O}$ is just the component of $\wp_\Lambda^{-1}(V)$ contained in Ω_u. Since by hypothesis F_o is forward invariant and $\overline{F_o} \subset V$, then $F_o \subset \mathcal{O}$ and therefore $\mathcal{P}$ maps $\mathcal{O}$ onto V by a ramified two-fold covering by Corollary 4.1.

Then Definition 4.1 and Proposition 4.2 tell us that $(\mathcal{P} : \mathcal{O}, \mathcal{P}(\mathcal{O}))$ is quadratic-like. Theorem 4.2 gives that $\partial K(\mathcal{P})$ is a quasicircle. □

There are many lattices for which the associated function $\wp_\Lambda$ satisfies the hypotheses of Theorem 4.3. We prove the existence of some examples here.

4.1. **Examples of quadratic-like elliptic functions.**

Theorem 4.4. *Suppose Λ is a real rectangular square lattice or a real triangular lattice with $g_3 > 0$, and suppose that $\wp_\Lambda$ has a real attracting fixed point p_o such that the forward invariant Fatou component F_{p_o} contains at most one critical value. Then for each forward invariant Fatou component F of $\wp_\Lambda$ there is an open set G and a period parallelogram Ω of Λ such that*

$$\overline{F} \subset G \subset \Omega.$$

Proof. Assume first that $\Lambda = [\lambda, \lambda i]$ is rectangular square for some $\lambda > 0$. By [14] (Proposition 6.7.2), the attracting fixed point $p_o > 0$ is the only non-repelling cycle and we have that the immediate attracting basin $F_{p_o} = F$ contains the positive critical value $e_1 > 0$ which satisfies:

$$0 \leq n_o\lambda < e_1 \leq p_o < (n_o + 1)\lambda$$

for some nonnegative integer n_o. Since $e_3 = 0$ is a pole, it is in the Julia set; by hypothesis $e_2 \notin F$. We claim therefore that F intersects exactly two quadrants in the plane: I and IV. If F were to intersect quadrant II or III, then by symmetry with respect to the origin and each axis, $e_2 = -e_1 \in F$, a contradiction. Furthermore F cannot intersect the imaginary axis because it is open and would therefore intersect quadrant II, and by symmetry, III. By the symmetry arising from the periodicity of $\wp_\Lambda$, it follows that F does not intersect any lines of the form $\Re(z) = m\lambda$, $m \in \mathbb{Z}$ (the vertical lines at lattice points).

Moreover we claim that F cannot intersect any line of the form $\Im(z) = (2m + 1)\lambda/2$, $m \in \mathbb{Z}$ (the horizontal half lattice lines). By periodicity of $\wp_\Lambda$ it is enough to consider points of the form $z = a + \frac{\lambda}{2}i$ for $0 \leq a \leq \frac{\lambda}{2}$. But it is well-known (cf. [12]) that for a rectangular square lattice, the horizontal line segment from $0 + \frac{\lambda}{2}i$ to $\frac{\lambda}{2} + \frac{\lambda}{2}i$ maps under $\wp_\Lambda$ onto the horizontal line segment from $-e_1$ to 0 along the negative real axis. In other words, its image does not lie in quadrants I and IV, so these lines form boundary lines for F.

In particular, we have just shown that F is completely contained in the period square Ω with vertices (going clockwise around the square):

$$n_o\lambda + (\lambda/2)i, (n_o + 1)\lambda + (\lambda/2)i, (n_o + 1)\lambda - (\lambda/2)i, n_o\lambda - (\lambda/2)i.$$

This is illustrated in Figure 6 for $n_o = 0$. By [12] and [14] (Lemma 4.4), we know that the lines with rectangular equations $y = x$ and all parallel lines of the form $y = x+m\lambda$, $m \in \mathbb{Z}$, all map to the nonnegative imaginary axis, and lines $y = -x$, $y = -x + m\lambda$, $m \in \mathbb{Z}$, map to the non-positive imaginary axis. The intersection of any two lines of this form are precisely the critical points of the form: $z = (m/2)(\lambda + i\lambda)$, $m \in \mathbb{Z}$; we know that these map to the origin or are poles, depending on whether m is even or odd.

We therefore consider the open square S bounded by the lines $y = x$, $y = -x$, $y = \lambda - x$, $y = -\lambda + x$, and its translation $S_{n_o} = S + n_o\lambda \subset \Omega$. Each square $S_m, m \in \mathbb{Z}$ maps two-to-one onto the open right half plane (quadrants I and IV) except at the one critical point in S_m; by our construction, ∂S_m maps onto the imaginary axis and the point at ∞.

We claim that $\overline{F} \subset S_{n_o} \subset \Omega$. We have already shown that $F \subset S_{n_o} \subset \Omega$. If $\overline{F}$ contains a point in ∂S_{n_o}, then by continuity of $\wp_\Lambda$ and forward invariance of F, F contains points arbitrarily close to the line $\Re(z) = n_o\lambda$, which we have shown cannot happen. Since F is bounded, it must be a bounded distance from all poles (lattice points) and prepoles, which are the vertices of S_{n_o}. Therefore the result holds for square lattices using $G = S_{n_o}$.

We now turn to the case of a real triangular lattice with real attracting fixed point p_o. We argue as in the square lattice case; the immediate attracting basin F_{p_o} lies in quadrants I and IV. In the triangular case we have symmetry of the Fatou set about each axis, origin, and with respect to rotation through $2\pi/3$ radians. Furthermore our assumptions and Proposition 3.5 imply that there are three disjoint forward invariant Fatou components; from this we can conclude that $F = F_{p_o}$ cannot cross the imaginary axis. By Proposition 3.1 we can choose as generators of Λ the vector $\lambda e^{\pi i/3}$ and its conjugate; let Ω be the period parallelogram containing $e_1 \in F$. There is a smaller interior region S in Ω consisting of points sent into quadrants I and IV. Its description is more complicated than for the square case (cf. [12] and Figure 7), but its boundary is an analytic curve. The rest of the proof is the same as for the square case. Moreover, there are two other forward invariant

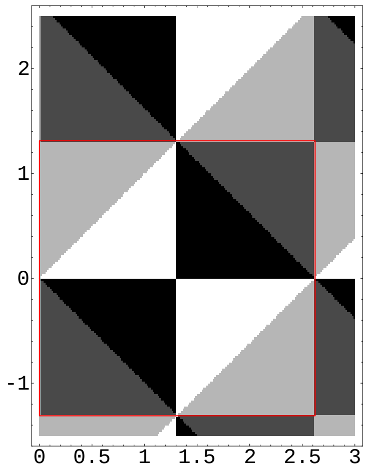

FIGURE 6. Quadrant preimages for a real square lattice

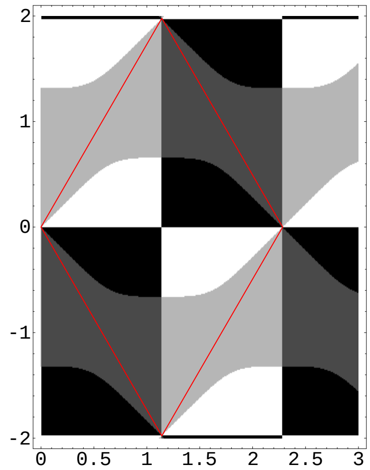

FIGURE 7. Quadrant preimages for a real triangular lattice

components of $F(\wp_\Lambda)$, namely $e^{2\pi i/3}F$ and $e^{4\pi i/3}F$, and the result holds for these by symmetry.

□

In Figures 7 and 6 we show the regions of a period parallelogram which get mapped under $\wp_\Lambda$ to Quadrants I (the white points) andIV (the black points)for square and triangular lattices respectively. The points colored gray get mapped onto the remaining two quadrants.

Theorem 4.5. *Suppose Λ is a real rectangular square lattice or a real triangular lattice with $g_3 > 0$, and suppose that $\wp_\Lambda$ has a real attracting fixed point p_o such that the forward invariant Fatou component F_{p_o} contains at most one critical value. Then $\wp_\Lambda$ restricted to the interior of the (appropriately chosen) period parallelogram containing p_o is quadratic-like.*

Proof. We apply Theorems 4.4 and 4.3. □

Corollary 4.6. Under the hypotheses of Theorem 4.5, $J(\wp_\Lambda)$ is locally connected.

Proof. The hypotheses imply that $F(\wp_\Lambda)$ consists of a countable union of simply connected open disks whose boundaries are quasicircles. The result follows from Theorem 4.3, and the fact that quasicircles are locally connected.

□

Corollary 4.7. Suppose Λ is a real square lattice or a real triangular lattice with $g_3 > 0$, and $\wp_\Lambda$ has a superattracting fixed point. Then $\wp_\Lambda$ is quadratic-like and $J(\wp_\Lambda)$ is locally connected.

5. Examples of locally Sierpinski Julia sets

5.1. **Triangular lattices.** The space of Weierstrass elliptic $\wp$ functions with triangular period lattices was studied by the first author and Koss in [15]. It was shown there that there the hyperbolic parameters are quasiconformally stable, so Corollary 4.7 is true much more generally.

Theorem 5.1. *If Λ is a real triangular lattice with $g_3 > 0$, and $\wp_\Lambda$ has an attracting fixed point, then $J(\wp_\Lambda)$ is a Sierpinski curve on $\mathbb{C}_\infty$ and $J(\wp_\Lambda)$ is locally Sierpinski.*

Proof. Suppose we have proved that $J(\wp_\Lambda)$ is a Sierpinski curve on $\mathbb{C}_\infty$; we now consider generators of Γ of the form $[\gamma, e^{2\pi i/3}\gamma]$, with $\gamma \in \mathbb{R}$, and use these to form the boundary of a period parallelogram Q. By Proposition 5.6 of [16], we have that the intersection of $J(\wp_\Lambda)$ with the boundary of (this particular) Q is a Cantor set. Setting $J_Q = J(\wp_\Lambda) \cap Q$, it follows from our verification of the properties in Definition 1.1 that the outer curve of J_Q is a simple closed curve homeomorphic to the graph of the Cantor function (instead of pieces of intervals between Cantor set points we have pieces of quasicircles). We then apply Remark 1.1 to the period parallelogram J_Q to obtain a locally Sierpinski curve.

We now turn to the proof that $J(\wp_\Lambda)$ is a Sierpinski curve on $\mathbb{C}_\infty$; we show the conditions of Definition 1.1 are satisfied. By Theorem 3.6 and Corollary 4.6 we have that $J(\wp_\Lambda)$ is connected and locally connected. Since the Julia set is closed and is not all of $\mathbb{C}_\infty$ we know that it is nowhere dense and compact on $\mathbb{C}_\infty$. It remains to check Property (5) of Definition 1.1.

By Corollary 3.5 there are three attracting fixed points; let us denote the attracting fixed points by p_1, p_2, p_3, and each immediate basin of attraction by A_1, A_2, and A_3. Since p_j is an attracting fixed point and Λ is a triangular lattice, we know that no other critical value tends to p_j apart from e_j. Therefore, the Fatou set of $\wp_\Lambda$ consists of A_j, $j = 1, 2, 3$ and all of the preimages.

We only need to check that the boundaries of pairwise disjoint complementary regions of $J(\wp_\Lambda)$ are disjoint simple closed curves. The boundaries of complementary regions of $J(\wp_\Lambda)$ are precisely the boundaries of the preimages of the A_j's. If ∂A_j is a simple closed curve, our hypothesis implies that its preimages will be simple closed curves as well.

By Theorem 4.5 we are guaranteed that Λ satisfies the hypotheses of Theorem 4.3. Therefore, by the theory of polynomial-like mappings we know that the boundary of A_j is a quasicircle and therefore a simple closed curve.

Finally we need to show that disjoint Fatou components have disjoint boundaries. Any Fatou component of $\wp_\Lambda$ is a component of $\wp_\Lambda^n(A_j)$ for some $n \leq 0$ and some $j = 1, 2, 3$. Let W_1 and W_2 be two distinct Fatou components and assume that there is a point $q \in \partial W_1 \cap \partial W_2$. There are two cases:

Case 1: Both components terminate at the same fixed component A_j. Then there exists an integer $k > 0$ such that $\wp_\Lambda^k(W_1) = \wp_\Lambda^k(W_2)$ while $\wp_\Lambda^{k-1}(W_1) \neq \wp_\Lambda^{k-1}(W_2)$; i.e., $\wp_\Lambda$ takes $\wp_\Lambda^{k-1}(W_1)$ and $\wp_\Lambda^{k-1}(W_2)$, two disjoint open sets, onto the same set. Further, $\wp_\Lambda^{k-1}(q) \in \partial\wp_\Lambda^{k-1}(W_1) \cap \wp_\Lambda^{k-1}(W_2)$. Hence, $\wp_\Lambda^{k-1}(q)$ must be a critical point for $\wp_\Lambda$. But if a critical point is in $\partial\wp_\Lambda^k(W_1)$, then we have a critical value in $\partial\wp_\Lambda^{k+1}(W_1)$. This is not possible, however, since $\partial\wp_\Lambda^{k+1}(W_1) \subset J(\wp_\Lambda)$ and we know that none of our critical values are in the Julia set since $\wp_\Lambda$ is hyperbolic.

Case 2: W_1 and W_2 terminate at different fixed components, say A_1 and A_2 respectively. We note that we can relabel sets if necessary so that $e^{2\pi i/3}A_1 = A_2$. Then there exists an integer $k > 0$ such that $e^{2\pi i/3}\wp_\Lambda^k(W_1) = \wp_\Lambda^k(W_2)$ while $e^{2\pi i/3}\wp_\Lambda^{k-1}(W_1) \neq \wp_\Lambda^{k-1}(W_2)$. Then $q \in \partial W_1 \cap \partial W_2$ implies that $\wp_\Lambda^k(q) \in A_1 \cap A_2$. However it follows from the proof of Theorem 4.4 that $\overline{A_1} \cap \overline{A_2} = \emptyset$, so no point q exists.

Therefore, there is no point on the boundary of two pairwise disjoint complementary regions of $J(\wp_\Lambda)$.

□

5.2. **Nonhyperbolic examples: square lattices.** We discuss the case of square lattices here; the argument is basically the same as for the previous example with some small changes. We begin with a general result about the dynamics of $\wp_\Lambda$ when Λ is a square lattice.

Proposition 5.1. Suppose Λ is a square lattice and is generated by λ and λi for some nonzero $\lambda \in \mathbb{C}$, and let $\wp_\Lambda$ denote the Weierstrass elliptic $\wp$ function with period lattice Λ.

(1) $g_3 = 0$; therefore the possible postcritical values are $e_3 = 0, e_1 = \sqrt{g_2}/2$, and $e_2 = -e_1$, where g_2 is a nonzero complex number;

(2) $P(\wp_\Lambda)$ includes ∞, and $e_3 \in J(\wp_\Lambda)$. Furthermore, since $e_1 = -e_2$, and $\wp_\Lambda$ is even, $P(\wp_\Lambda) = \overline{\bigcup_{n\geq 0} \wp_\Lambda^n(e_1) \cup \{e_2, 0, \infty\}}$, (so is completely determined by the forward orbit of e_1).
(3) There is at most one attracting orbit;
(4) if z_o is a superattracting fixed point for $\wp_\Lambda$ then $J(\wp_\Lambda)$ is connected;
(5) For every lattice $\Lambda_o = [\lambda_o, \lambda_o i]$ $\lambda_o \in \mathbb{C}$ with a superattracting fixed point z_o, there is a neighborhood U of λ_o, $U \subset \mathbb{C}$ such that for every $\lambda \in U$, the associated function $\wp_\Lambda$ has an attracting fixed point $z(\lambda)$, there is exactly one critical value and one critical point in its immediate attracting basin, $A(\lambda)$, and $A(\lambda)$ is simply connected. Hence $J(\wp_\Lambda)$ is connected for all $\Lambda \in U$.

Proof. Properties 1,2 and 3 were shown in [14] and [15]. Property 4 was also shown in [14] and is subsumed by 5. To show 5, suppose z_o is a superattracting fixed point of $\wp_{\Lambda_o}$. Since the fixed point and its derivative vary continuously (even analytically) in λ, nearby all maps have attracting fixed points. In [16], Proposition 6.3, it was proved that the Julia set moves continuously as λ varies in this setting. The result then follows from (4).

□

We can now prove the following.

Theorem 5.2. *If Λ is a real rectangular square lattice and $\wp_\Lambda$ has an attracting fixed point, then $J(\wp_\Lambda)$ is a Sierpinski curve on $\mathbb{C}_\infty$ and $J(\wp_\Lambda)$ is locally Sierpinski.*

Proof. We denote the attracting fixed point by p and its immediate basin of attraction by A_p. Since p is an attracting fixed point and Λ is a square lattice, by Proposition 5.1 we know that two of the critical values tend to p; the third critical value is a pole. Therefore, the Fatou set of $\wp_\Lambda$ consists of A_p and all of its preimages. Since p is attracting we know that A_p is simply connected and that $\overline{A_p} \subset \Omega_u$ where Ω_u is a period square as in Theorem 4.3. Having all of our critical values accounted for with $e_1 \in A_p$ and $e_2 = -e_1$ strictly inside one of the (disjoint) preimages of A_p not containing p, we know that all of the preimages of the A_p are also simply connected. Hence, $J(\wp_\Lambda)$ is the Riemann sphere minus a countable number of simply connected open sets. Therefore $J(\wp_\Lambda)$ is connected. The local connectivity of $J(\wp_\Lambda)$ follows from the fact that it is quadratic-like near the fixed point.

Since the Julia set is closed and is not all of $\mathbb{C}_\infty$ we know that it is nowhere dense and compact on $\mathbb{C}_\infty$. Therefore, we only need to check that the boundaries of pairwise disjoint complementary regions of $J(\wp_\Lambda)$ are disjoint simple closed curves.

The boundaries of complementary regions of $J(\wp_\Lambda)$ are precisely the boundaries of the preimages of A_p. To show that all of these boundaries are simple closed curves, it suffices to show that ∂A_p is a simple closed curve. This follows since all critical points are accounted for, and therefore if ∂A_p is a simple closed curve, this will guarantee that its preimages are simple closed curves.

By Theorem 4.4 we have that Λ satisfies the hypotheses of Proposition 4.3. Therefore, by the theory of polynomial-like mappings we know that the boundary of A_p is a quasicircle and therefore a simple closed curve.

Finally we need to show that disjoint Fatou components have disjoint boundaries. Any Fatou component of $\wp_\Lambda$ is a component of $\wp_\Lambda^n(A_p)$ for some $n \leq 0$. Let W_1 and W_2 be two Fatou components and assume that there is a point $q \in \partial W_1 \cap \partial W_2$. There exists some $k > 0$ such that $\wp_\Lambda^k(W_1) = \wp_\Lambda^k(W_2)$ while $\wp_\Lambda^{k-1}(W_1) \neq \wp_\Lambda^{k-1}(W_2)$. We have $\wp_\Lambda$ taking $\wp_\Lambda^{k-1}(W_1)$ and $\wp_\Lambda^{k-1}(W_2)$, two disjoint open sets, onto the same set. Further, $\wp_\Lambda^{k-1}(q) \in \partial\wp_\Lambda^{k-1}(W_1) \cap \wp_\Lambda^{k-1}(W_2)$. Hence, $\wp_\Lambda^{k-1}(q)$ must be a critical point for $\wp_\Lambda$. But if a critical point is in $\partial\wp_\Lambda^k(W_1)$, then we have a critical value in $\partial\wp_\Lambda^{k+1}(W_1)$. This is not possible, however, since $\partial\wp_\Lambda^{k+1}(W_1) \subset J(\wp_\Lambda)$ and we know that the only critical point in $J(\wp_\Lambda)$ is a pole; i.e., there are no critical values in $J(\wp_\Lambda) \cap \mathbb{C}$.

Therefore, we can have no point in the boundary of two pairwise disjoint complementary regions of $J(\wp_\Lambda)$ and it follows that $J(\wp_\Lambda)$ is a Sierpinski curve on $\mathbb{C}_\infty$.

We show that $J(\wp_\Lambda)$ is locally Sierpinski as in the previous result. We consider generators of the form $[\gamma, \gamma i]$ for the lattice, with $\gamma > 0$. We then apply Proposition 5.5 of [16] to see that forming the square period parallelogram Q with vertices $0, \gamma, \gamma i$, and $\gamma + \gamma i$, J_Q has a Cantor set intersection with ∂Q. Arguing as in Theorem 5.1 we have a locally Sierpinski Julia set.

□

We show a typical locally Sierpinski carpet tile in Figure 8.

5.3. **Constructing the Julia set for a square lattice with a superattracting fixed point.** Suppose we have a square lattice which has a superattracting fixed point, or we are at a lattice which is nearby so we have an attracting fixed point. Then we have a simply connected immediate attracting basin, A_o, in some period parallelogram Q.

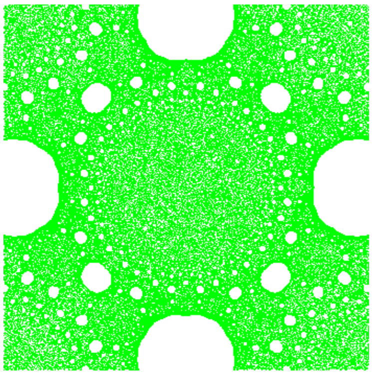

FIGURE 8. A single Sierpinski carpet tile for a square lattice

The preimage of A_o in Q is just itself and there is a copy of it, $A_o + \omega, \omega \in \Lambda$ in each period parallelogram. (Since there is a single critical point in A_o it maps two-to-one onto itself in Q.)

Therefore the set $\wp_\Lambda^{-2}(A_o)$ has infinitely many simply connected sets in Q; the two largest ones are the preimages of A_o (see Figures 9 and 10), and as $|\omega|$ gets larger so the the set $A_o + \omega$ gets closer to ∞, its preimage in Q is smaller (near the pole at 0). This repeats indefinitely until we get:

$$J(\wp_\Lambda) = \mathbb{C}_\infty \setminus \cup_{j=0}^{\infty} \wp_\Lambda^{-j} A_o,$$

which is illustrated in Figures 11 and 12.

5.4. **Sierpinski curves on the torus and planar $\mathcal{S}$-curves.** Suppose we have a lattice Λ for which we have a locally Sierpinski Julia set resulting from an attracting fixed point as above. We let F_o denote a forward invariant component of $F(\wp_\Lambda)$ corresponding to an attracting fixed point p_o. By mapping the fixed point p_o to ∞ via a Mobius transformation, we send the region F_o to a neighborhood of ∞. The boundary of F_o is a simple closed curve as shown above. We can then apply Remark 1.1 to obtain a planar $\mathcal{S}$-curve. This is illustrated in Figures 13 (obtained from a square lattice) and 14. In Figure 14 the Julia set is green, while the complementary regions are colored shades of blue according to which of the three attracting fixed points each Fatou component is attracted.

Furthermore, if we identify fundamental regions for $\wp_\Lambda$, we can view $J(\wp_\Lambda)$ on $\mathbb{C}/\Lambda$, a torus, and for the examples above we have a Sierpinski curve on T^2.

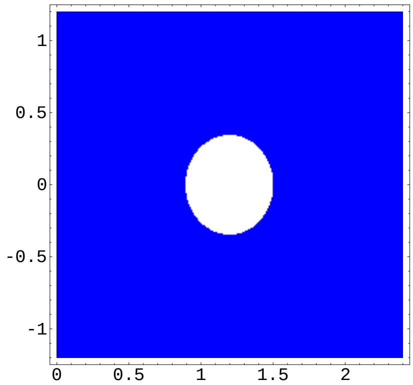

Figure 9. A piece of the immediate attracting basin of a fixed point for a square lattice

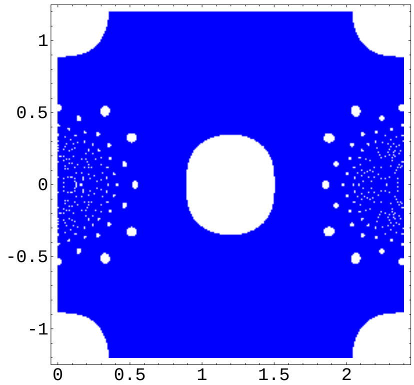

Figure 10. Points sent to the region after two iterations

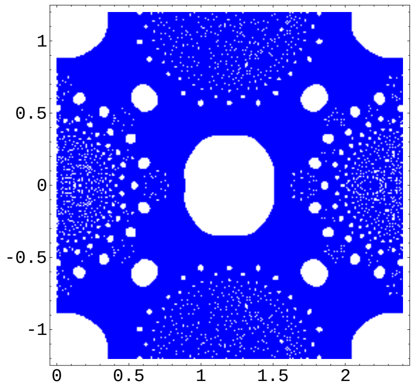

Figure 11. After 3 iterations

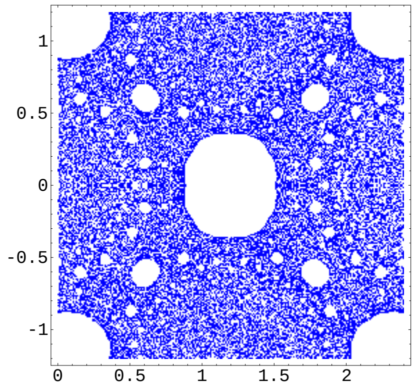

Figure 12. The Julia set after 10 iterations

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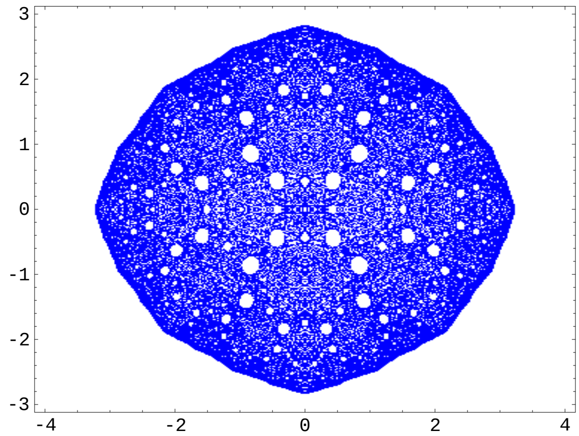

FIGURE 13. $J(\wp_\Lambda)$ conjugated by a Mobius transformation

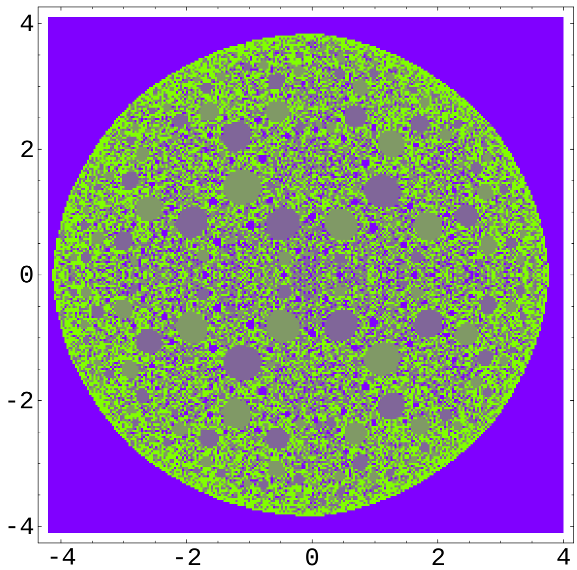

FIGURE 14. A conjugate copy $J(\wp_\Lambda)$ for a triangular lattice

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