Homework - Probability

1. Suppose that a bag contains 7 black balls, 6 yellow balls, 4 green balls, and 3 red balls. You shake the bag well, and remove one ball without looking into the bag.

   (a) What is the probability that the ball you remove is red? Black? Yellow? Green? White?

   (b) What is the probability that the ball you pick is either black or green?

   (c) What is the probability that you have picked a ball whose color is not red?

2. A die is painted so that three sides are red, two sides are blue, and one side is green. Thus, rolling the die has three possible outcomes, R, B, and G.

   (a) What is the probability that the die will come up blue?

   (b) What is the probability that the die will not come up red?

   (c) What is the probability that the face showing is either red or blue?

3. The painted die from the previous problem is rolled twice. Denote the nine possible outcomes by RR, RB, etc.

   (a) Find the probability of each element of the sample space.

   (b) What is the probability that at least one roll will be red?

   (c) What is the probability that neither roll is blue?

   (d) What is the probability that the two rolls will have different colors?

4. The painted die from the previous problem is rolled twice. Use the addition rule to find the following probabilities.

   (a) The probability that either both rolls are red or both rolls are blue.

   (b) The probability that either both rolls are red or exactly one roll is blue.

   (c) The probability that either at least one roll is red or exactly one roll is blue.

   (d) The probability that either at least one roll is red or at least one roll is blue.

5. Suppose that a fair die is rolled twice.

   (a) Let A be the event that the first roll is ≥ 2. Let B be the event that the second roll is ≤ 4. Find $\mathbb{P}(A \text{ and } B)$, and prove that A and B are independent.

   (b) Let A be the event that the first roll is ≥ 2. Let B be the event that the sum of the rolls is ≤ 4. Find $\mathbb{P}(A \text{and } B)$, and prove that A and B are not independent. Why does this make sense?

6. Two events A and B are said to be **mutually exclusive** if the probability that they both occur is zero.

   (a) Suppose you roll a fair six-sided die. Give an example of two events that are mutually exclusive.

(b) Let A and B be events such that $\mathbb{P}(A) > 0$ and $\mathbb{P}(B) > 0$. Prove that it is impossible for A and B to be both independent and mutually exclusive.

7. Show that $\mathbb{P}(A \cap B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1$ for any two events A and B.

8. Consider the experiment of flipping a fair coin twice. Let X be the number of heads minus the number of tails.

   (a) Find the possible values of X.

   (b) Find the probability mass density of X.

   (c) Find the expected value of X.

9. Consider the experiment of flipping a biased coin (which comes up heads with probability $\frac{3}{4}$) twice. Let X be the number of heads minus the number of tails.

   (a) Find the possible values of X.

   (b) Find the probability mass density of X.

   (c) Find the expected value of X.

10. Consider the experiment of flipping a fair coin three times. Let X be the square of the number of heads.

    (a) Find the possible values of X.

    (b) Find the probability mass density of X.

    (c) Find the expected value of X.

11. An encyclopedia salesman visits three customers each day, and with each he has a probability of $\frac{1}{4}$ of making a sale. For each sale he earns a commission of \$100 and if he makes three sales in one day, he earns a \$50 bonus from his company. Let X be his daily earnings. What is the probability mass density of X?

12. You own one share of stock for two years, and each year the value of the stock changes by $+2$, $+1$, 0, -1, each with probability $\frac{1}{4}$. Suppose that the changes in the two years are independent.

    (a) Find the possible values of X.

    (b) Find the probability mass density of X.

    (c) Find the expected value of X.

13. Consider the following game played at a casino. A player bets on one of the numbers 1 through 6. Three dice are then rolled, and if the number bet by the player appears i times, for $i = 1, 2, 3$, then the player wins i dollars; on the other hand, if the number bet by the player does not appear on any of the dice, then the player loses 1 dollar. Is the game fair? If the game is not fair, who has the advantage, the player or the casino?

14. Consider the experiment of flipping a biased coin (which comes up heads with probability $\frac{3}{5}$) four times. Find the expected value of each of the following random variables.

    (a) X is the number of heads.

    (b) Y is the number of heads minus the number of tails.

    (c) Z is equal to $|Y|$

15. A box contains 5 marbles; 2 are labeled with the number 1 and 3 are labeled with the number 2. Suppose you reach in and select two marbles, without replacement. Let X be the product of the two numbers drawn. Find $\mathbb{E}[X]$.