Problem 2.5.5. Show that $x^2 - 1, x^2 + 1, x + 1$ are linearly independent on $\mathbb{R}$.

Solution. Note that

$$w(x^2 - 1, x^2 + 1, x + 1)_{x=0} = \begin{vmatrix} x^2 - 1 & x^2 + 1 & x + 1 \\ 2x & 2x & 1 \\ 2 & 2 & 0 \end{vmatrix} = \begin{vmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 2 & 2 & 0 \end{vmatrix} = 4 \neq 0.$$  

Theorem 2.15 in the book then implies that $x^2 - 1, x^2 + 1, x + 1$ are linearly independent on $\mathbb{R}$.

Problem 2.5.6. Show that $e^x, e^{2x}, e^{3x}$ are linearly independent on $\mathbb{R}$.

Solution. Note that

$$w(e^x, e^{2x}, e^{3x})_{x=0} = \begin{vmatrix} e^x & e^x & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = 2 \neq 0.$$  

Theorem 2.15 in the book then implies that $e^x, e^{2x}, e^{3x}$ are linearly independent on $\mathbb{R}$.

Problem 2.5.7. Show that $e^{4x}, xe^{4x}, x^2 e^{4x}$ are linearly independent on $\mathbb{R}$.

Solution. Note that

$$w(e^{4x}, xe^{4x}, x^2 e^{4x})_{x=0} = \begin{vmatrix} e^{4x} & xe^{4x} & x^2 e^{4x} \\ 4e^{4x} & e^{4x} + 4xe^{4x} & 2xe^{4x} + 4x^2 e^{4x} \\ 16e^{4x} & 8e^{4x} + 16xe^{4x} & 2e^{4x} + 16xe^{4x} + 16x^2 e^{4x} \end{vmatrix}_{x=0} = \begin{vmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 8 & 2 \end{vmatrix} = 2 \neq 0.$$  

Theorem 2.15 in the book then implies that $e^{4x}, xe^{4x}, x^2 e^{4x}$ are linearly independent on $\mathbb{R}$.

Problem 2.5.8. Show that $e^x, e^x \cos x, e^x \sin x$ are linearly independent on $\mathbb{R}$.

Solution. Note that

$$w(e^x, e^x \cos x, e^x \sin x)_{x=0} = \begin{vmatrix} e^x & e^x \cos x & e^x \sin x \\ e^x & e^x (\cos x - \sin x) & e^x (\sin x + \cos x) \\ e^x & -2e^x \sin x & 2e^x \cos x \end{vmatrix}_{x=0} = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = 1.$$
Theorem 2.15 in the book then implies that $e^x, e^x \cos x, e^x \sin x$ are linearly independent on $\mathbb{R}$. □

**Problem 2.5.12.** Show that $\sin^2 x, \cos^2 x, \cos 2x$ are linearly dependent on $\mathbb{R}$.

**Solution.** Using the trigonometric identities
\[
\sin^2 x + \cos^2 x = 1
\]
\[
\cos 2x = 1 - 2 \sin^2 x,
\]
we see that
\[
\sin^2 x - \cos^2 x + \cos 2x = (2 \sin^2 x - \sin^2 x) - \cos^2 x + 1 - 2 \sin^2 x
\]
\[
= -(\sin^2 x + \cos^2 x) + 1
\]
\[
= -1 + 1
\]
\[
= 0.
\]
Hence $\sin^2 x, \cos^2 x, \cos 2x$ are linearly dependent on $\mathbb{R}$. □

**Note.** We cannot use the Wronskian here because there are linearly independent collections with nonzero Wronskians (see page 108 example 2).

**Problem 3.1.1.** Determine if $y = e^x$ solves the differential equation
\[
y'' - 2y' + y = 0.
\]
If $y = e^x$ is a solution to (1), then determine if $y = e^x$ satisfies the initial condition $y(0) = -1$.

**Solution.** Note that for $y = e^x$, $y' = y'' = e^x$. So, for $y = e^x$, we have
\[
y'' - 2y' + y = e^x - 2e^x + e^x = 0.
\]
Hence $y = e^x$ solves (1). Obviously, $e^0 = 1 \neq -1$, so $y = e^x$ does not satisfy the initial condition $y(0) = -1$. □
Problem 3.1.18. Consider the differential equation
\[ y' = xy - xy^3 = xy (1 - y) (1 + y). \]

(a) Determine the equilibrium solutions to (2).
(b) On each region determined by the equilibrium solutions, determine when the graph is increasing, decreasing, concave up, and concave down.

Solution. (a) The equilibrium solutions to (2) are the constants \( y \) such that
\[ xy (1 - y) (1 + y) = 0 \]
for all \( x \). Dividing through by \( x \) gives
\[ y (1 - y) (1 + y) = 0. \]
Hence the equilibrium solutions are \( y = 0 \) and \( y = \pm 1 \).
(b) See Figure 3.1.

\[ \square \]

Figure 3.1. The derivative behavior and concavity to (2).