My Curvature Conventions

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1 Connection on a Vector Bundle

Let M be a smooth manifold, and let E be an $\mathbb{R}^n$-vector bundle over M, with a connection D. Let $\mu_1, \cdots, \mu_n$ be local frame over an open subset $U \subset M$, and regard $(\mu_1, \cdots, \mu_n)$ as **row** vector at each point of U, we have the Christoffel symbols of the connection D defined as:

$$D_{\frac{\partial}{\partial x^i}} \mu_j =: \Gamma_{ij}^k \mu_k.^*$$

For a section s, it can be written locally as $s|_U = \sum_j a^j \mu_j = (\mu_1, \cdots, \mu_n) \cdot (a^1, \cdots, a^n)^t$.Thus

$$D(a^j \mu_j) = (da^j)\mu_j + a^j A \mu_j,$$

where A is a connection matrix with entries 1-forms, defined as $A_j^k := \Gamma_{ij}^k dx^i$, and $A\mu_j = \mu_k A_j^k$. Therefore

$$D\mu_j = \Gamma_{ij}^k dx^i \otimes \mu_k = \mu_k A_j^k.$$

Note that each $A_j^k \in T^*M|_U$, with $A_{\text{column}}^{\text{row}}$. We can then write a connection D as $D = d + A$. For $s|_U = \sum_j a^j \mu_j$, we have

$$D(a^j \mu_j) = (\mu_1, \cdots, \mu_n) \cdot \Big((da^1, \cdots, da^n)^t + A \cdot (a^1, \cdots, a^n)^t \Big),$$

i.e., A acts on the **left** on the coefficient column vector $(a^1, \cdots, a^n)$.

2 Connection on Endomorphism Bundle

With this notation, we can verify that using the induced connection $\widetilde{D}$ on $\mathrm{End}(E) \cong E \otimes E^*$ is given by $\widetilde{D} = d + [A, \cdot]$, where A is the connection matrix of D. Let

*Note that Chen defines connection matrix first, then defines Γ_{ij}^k, but I will do the converse.

$\sigma := \sigma^i_j \mu_i \otimes \mu_j$ be a local section of $\mathrm{End}(E)$, where $\{\mu^*_j\}$ is the dual frame. We compute:

$$D(\sigma^i_j \mu_i \otimes \mu^*_j) = (d\sigma^i_j)\mu_i \otimes \mu^*_j + \sigma^i_j A^k_i \mu_k \otimes \mu^*_j + \sigma^i_j \mu_i \otimes ((A^*)^k_j \mu^*_j), \tag{1}$$

where A^* is a connection matrix of the induced connection on E^*. Note that

$$0 = d(\mu_i, \mu^*_j) = (D\mu_i, \mu^*_j) + (\mu_i, D^*\mu^*_j) = A^j_i + (A^*)^i_j,$$

hence $A^* = -A^t$. Therefore (1) gives:

$$\begin{aligned} D(\sigma^i_j \mu_i \otimes \mu^*_j) &= \big(d\sigma^i_j + \sigma^k_j A^i_k - \sigma^i_k A^k_j\big)\mu_i \otimes \mu^*_j \\ &= \big(d\sigma^i_j + (A \cdot \sigma)^i_j - (\sigma \cdot A)^i_j\big)\mu_i \otimes \mu^*_j \\ &= d\sigma + [A, \sigma]. \end{aligned}$$

3 Curvature of Vector Bundle

We define the curvature of a connection D to be $F := D \circ D : \Omega(E) \longrightarrow \Omega^2(E)$. For $\mu \in \Gamma(E)$, we have

$$F(\mu) = (d + A) \circ (d + A)(\mu) = (d + A)(d\mu + A \cdot \mu) = (dA) \cdot \mu + (A \wedge A) \cdot \mu.$$

More explicitly, suppose $(\mu_1, \cdots, \mu_n)$ is a local frame, and $\mu := a^j \mu_j$ as above, we have:

$$\begin{aligned} F(a^j \mu_j) &= (d + A)\Big((da^j)\mu_j + A^j_k a^k \mu_j\Big) \\ &= (d \circ da^j)\mu_j + A^k_j (da^j)\mu_k + d(A^j_k a^k)\mu_j + A^l_j \wedge A^j_k a^k \mu_l \\ &= A^k_j (da^j)\mu_k + (dA^j_k) a^k \mu_j - A^j_k \wedge (da^k)\mu_j + A^l_j \wedge A^j_k a^k \mu_l \\ &= \Big((dA^j_k) a^k + (A \wedge A)^j_k a^k\Big)\mu_j. \end{aligned}$$

Thus $F = dA + A \wedge A$, acting on the coefficient column from the **left**, and on the local frame on the **right**. If we write $A = A_i dx^i$, i.e., $(A_i)^k_j = \Gamma^k_{ij}$, then

$$\begin{aligned} F &= d(A_i dx^i) + (A_i dx^i) \wedge (A_j dx^j) \\ &= \frac{\partial A_i}{\partial x^j} dx^j \wedge dx^i + A_i \cdot A_j dx_i \wedge dx^j \\ &= \frac{1}{2}\left(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_j}{\partial x^i} + [A_i, A_j]\right) dx^i \wedge dx^j. \end{aligned}$$

Notice that F is defined locally, but it turns out to be a global object. F can be regarded as a section of $\Omega^2(\mathrm{End}(E))$, that is F is a endomorphism bundle-valued 2-form, or from another perspective, a matrix with entries 2-forms. From section 2, we have an induced connection $\widetilde{D}$ on the endomorphism bundle, hence we can compute $\widetilde{D}F$ as follows:

$$\begin{aligned}\widetilde{D}F &= dF + [A, F]\\ &= d(A \wedge A) + [A, dA + A \wedge A]\\ &= 0.\end{aligned}$$

That is, $\widetilde{D}F = 0$, i.e., $dF = [F, A]$. This is called the (second) Bianchi identity.

We now want to express F in terms of the Christoffel symbols. We first define the curvature operator R considered as an element of $\Omega^2(\mathrm{End}(E))$ as follows:

$$F : \Omega(E) \longrightarrow \Omega^2(E), \quad \mu \mapsto R(\cdot, \cdot)\mu,$$

and we define $R(\partial_i, \partial_j)\mu_l =: R^k_{lij}\mu_k$. Thus R^k_{lij} acts on μ_k from the right. We have:

$$R(\cdot, \cdot)\mu_l = F\mu_l = \frac{1}{2}\left(\frac{\partial \Gamma^k_{jl}}{\partial x^i} - \frac{\Gamma^k_{il}}{\partial x^j} + \Gamma^k_{im}\Gamma^m_{jl} - \Gamma^k_{jm}\Gamma^m_{il}\right) dx^i \wedge dx^j \otimes \mu_k.$$

Therefore

$$R^k_{lij} = \frac{\partial \Gamma^k_{jl}}{\partial x^i} - \frac{\Gamma^k_{il}}{\partial x^j} + \Gamma^k_{im}\Gamma^m_{jl} - \Gamma^k_{jm}\Gamma^m_{il}.$$

An important property that the curvature operator R satisfies is that

$$R(X, Y)\mu = [D_X, D_Y]\mu - D_{[X,Y]}\mu,$$

for all vector fields X, Y, and all section $\mu \in \Gamma(E)$.

4 Tangent Bundle Case

We now restrict our discussion to (M, g) a Riemannian manifold, and E $=TM$ is the tangent bundle, and D is the Levi-Civita connection ∇ on TM. Taking $\{\frac{\partial}{\partial x^i}\}$ to be our local frame, we have

$$R(\partial_i, \partial_j)\partial_l = R^k_{lij}\partial_k.$$

Using the Riemannian metric, we can define a new curvature tensor $R_{klij} := g_{km}R^m_{lij}$. Then it is easy to see

$$R_{klij} = \langle R(\partial_i, \partial_j)\partial_l, \partial_k\rangle.$$

The second Bianchi identity in the tangent bundle case is

$$R_{klij,h} + R_{lhij,k} + R_{hkij,l} = 0, \quad \forall k, l, i, j, h.$$

It is easy to verify $Riem := R_{klij} dx^i \otimes dx^j \otimes dx^l \otimes dx^k$ is a tensor field of type $(0,4)$ on M, called the Riemannian curvature tensor with respect to the metric g. We now define the associated intrinsic curvatures on (M, g).

1. The sectional curvature of the plane spanned by the linearly independent tangent vectors $X = a^i \partial_i|_p, Y = b^j \partial_j|_p \in T_pM$ is

$$\begin{aligned} K(X \wedge Y) &:= \frac{\langle R(X,Y)Y, X\rangle}{|X \wedge Y|^2} \\ &= \frac{\langle R(a^i\partial_i, b^j\partial_j)b^k\partial_k, a^l\partial_l\rangle}{g_{ik}g_{jl}(a^ia^kb^jb^l - a^ia^jb^kb^l)} \\ &= \frac{R_{lkij}a^ib^ja^lb^k}{g_{ik}g_{jl}(a^ia^kb^jb^l - a^ia^jb^kb^l)} \\ &= \frac{R_{ijkl}a^ib^ja^kb^l}{(g_{ik}g_{jl} - g_{ij}g_{kl})a^ib^ja^kb^l}, \end{aligned}$$

   where $|X \wedge Y|^2 := \langle X, X\rangle \cdot \langle Y, Y\rangle - \langle X, Y\rangle^2$.

2. The Ricci curvature in the direction $X = \sum_i a^i|_p \in T_pM$ is defined as

$$\begin{aligned} Ric(X, X) &:= g^{jl}\langle R(X, \partial_j)\partial_l, X\rangle \\ &= g^{jl}\langle R(a^i\partial_i, \partial_j)\partial_l, a^k\partial_k\rangle \\ &= a^ia^kg^{jl}R_{klij} \end{aligned}$$

   Hence $Ric(\partial_i, \partial_i) = g^{jl}R_{ilij}$. The Ricci tensor is $Ric_{ik} := R(\partial_i, \partial_k) = g^{jl}R_{klij} = g^{jl}R_{ijkl}$. That is the Ricci tensor is $Ric = R_{ik}dx^i \otimes dx^j$, which is of type $(0,2)$, while the Ricci curvature in a certain direction or along a vector field is a number at each point, hence is a smooth function on the manifold. We have $Ric_{ki} = g^{jl}R_{kjil} = g^{jl}R_{ilkj} = g^{lj}R_{ilkj} = R_{ik}$. Thus the Ricci tensor is a symmetric 2-tensor.

3. The scalar curvature is $R := g^{ik}Ric_{ik}$. Thus the Ricci curvature in the direction X is the average of the sectional curvature of all planes in T_pM containing X, and the scalar curvature is the average of the Ricci curvature of all unit vectors, i.e., of the sectional curvature of all planes in T_pM.