Constant Mean Curvature Surfaces in Asymptotically Flat Manifolds*

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1 Introduction: What is a CMC Surface

In this talk we will introduce a special type of surfaces called *constant mean curvature* (CMC) surfaces. This notion generalizes to any submanifolds with codimension 1 (called hypersurfaces), but we will first focus on 2-dimensional surfaces in the Euclidean space $\mathbb{R}^3$. This talk is mostly based on a series of lectures given by Lan-Hsuan Huang at the MSRI summer school (which the author attended) in 2012.
Let Σ^2 be a smooth surface in $\mathbb{R}^3$. For a point $p \in \Sigma$, consider all the curves in Σ passing through p. Now let κ_1 and κ_2 be the maximum and minimum curvature of such curves, respectively. Define $H := \kappa_1 + \kappa_2$. We can do this at every point, and thus we get a function $H : \Sigma \longrightarrow \mathbb{R}$, called the *mean curvature* of Σ. Thus Σ is called a *constant mean curvature* (CMC) surface if this function $H \equiv$ constant.

Example 1.1 (Helicoid)**.** *Let a and b be two constants, then consider the following parametrized surface:*

$$\varphi(u, v) = (u \cos v, u \sin v, av + b), \quad (u, v) \in \mathbb{R}^2. \tag{1.1}$$

This surface is called a Helicoid *(see figure 1), and is obtained by tracing out a line rotationally and vertically. It turns out that the mean curvature H is identically* 0 *in this case, which make a Helicoid a special case of a CMC surface, called a* minimal surface.

Example 1.2 (Cylinder)**.** *Let Σ be a cylinder with radius r (see figure 2). For any point p on Σ, we see that $\kappa_1 = \frac{1}{r}$ and $\kappa_2 = 0$ at p, thus $H \equiv \frac{1}{r}$. In particular, H is constant. Thus a cylinder is a CMC surface.*

Example 1.3 (Sphere)**.** *Let Σ be a round sphere S_r^2 with radius r in $\mathbb{R}^3$ (see figure 3). Then $\kappa_1 = \kappa_2 = \frac{1}{r}$, thus $H \equiv \frac{2}{r}$. Therefore S_r^2 is a CMC surface for each r fixed.*

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*The is the notes for the talk I gave at the graduate/faculty seminar at Duke.

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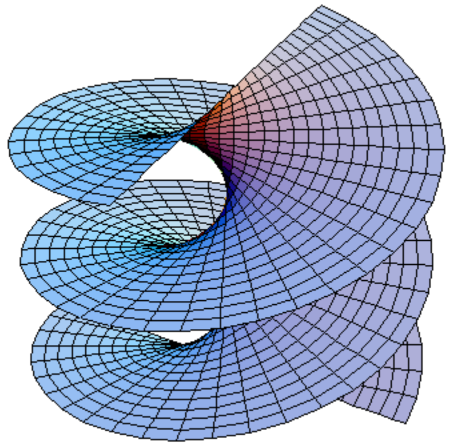

Figure 1: Helicoid, Picture taken from Wikipedia, `http://commons.wikimedia.org/wiki/File:Helicoid.PNG`

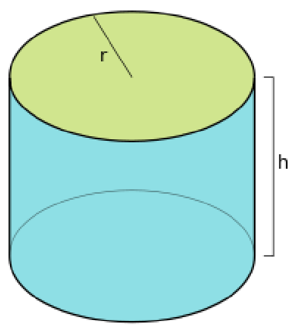

Figure 2: Cylinder, Picture taken from Wikipedia, `http://upload.wikimedia.org/wikipedia/commons/thumb/3/36/Circular_cylinder_rh.svg/200px-Circular_cylinder_rh.svg.png`

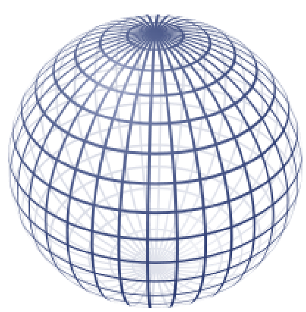

Figure 3: Sphere, Picture taken from Wikipedia, `http://upload.wikimedia.org/wikipedia/commons/thumb/7/7e/Sphere_wireframe_10deg_6r.svg/200px-Sphere_wireframe_10deg_6r.svg.png`

Now we briefly go over the history of studies of CMC surface ([6]):

1. 1841, Delaunay proved: the only CMC surfaces of revolution (rotation of a curve around an axis) are the plane, cylinder, sphere, catenoid, unduloid and nodoid.

2. 1853, A. D. Alexandrov proved: a compact embedded surface in $\mathbb{R}^3$ with constant *non-zero* mean curvature is a two sphere.

3. 1956, H. Hopf conjectured: any compact orientable immersed CMC hypersurface in $\mathbb{R}^n$ must be a round $n-1$ sphere.

4. 1982, Wu-Yi Hsiang constructed a counterexample in $\mathbb{R}^4$, disproving Hopf's conjecture.

5. 1984, Wente proved: there exists a CMC immersion of the 2 torus in $\mathbb{R}^3$.

6. 1984, Barbosa and do Carmo proved: any compact orientable immersed hypersurface in $\mathbb{R}^n$ that is *stable* and has constant *non-zero* mean curvature must be a round $n-1$ sphere.

7. 1996, Rugang Ye proved: In a strongly asymptotically flat manifold, there exists a surface near infinity that has prescribed constant mean curvature.

In this talk, we are interested in surfaces in an asymptotically flat manifold, which will be defined later. We first define mean curvature and related geometric quantities in general dimensions.

2 General CMC Hypersurfaces and Properties

Let (M^n, g) be a Riemannian manifold of dimension n, and let Σ^{n-1} be a closed (compact without boundary) hypersurface with the induced metric. Let $\overline{\nabla}$ be the Levi-Civita connection on M. Pick a point $p \in \Sigma$, let ν be a local normal vector field on Σ around p, then given any local tangent vector fields X and Y around p, we define

$$\mathrm{II}(X, Y)(p) := \langle \overline{\nabla}_X Y, \nu \rangle (p). \tag{2.1}$$

We thus get a globally defined symmetric $(0, 2)$-tensor A called the *second fundamental form.* Now the mean curvature of Σ is defined to be

$$H := \mathrm{tr}_g \mathrm{II}. \tag{2.2}$$

Thus Σ is called CMC if its mean curvature $H \equiv$ constant. We shall distinguish two types of CMC hypersurface: if $H \equiv 0$, then Σ is called a *minimal hypersurface*; otherwise we refer to Σ as a *non-zero* CMC hypersurface. Where do CMC hypersurfaces naturally come from? Roughly speaking, CMC hypersurfaces arise as critical points of variations of certain geometric quantities. Consider a variation of Σ along its normal direction ν in M with speed $\eta \in C^\infty(M)$. More precisely, we consider the following function:

$$F : \Sigma \times (-\delta, \delta) \longrightarrow M, \quad \delta > 0, \tag{2.3}$$

such that for $x \in \Sigma_t := F(\Sigma, t)$, and $-\delta < t < \delta$,

$$\begin{cases} \frac{\partial}{\partial t} F(x,t) = \eta(x,t)\nu(x,t), \\ F(\Sigma, 0) = \Sigma, \end{cases} \tag{2.4}$$

where $\nu(x,t)$ is the outward-pointing unit normal vector field along Σ_t.

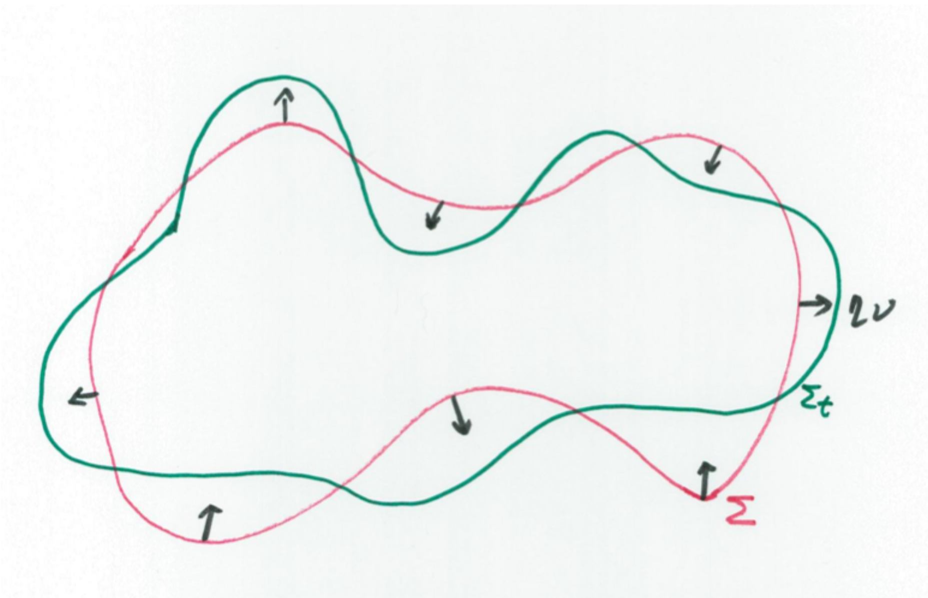

Figure 4: Variation of Σ along normal direction

Now let g_t be the induced metric on Σ_t, and $d\sigma_t, \mathrm{II}_t$ and H_t be the associated $(n-1)$-volume form, the second fundamental form and the mean curvature of Σ_t, respectively. We let

$$A_t := \int_{\Sigma_t} d\sigma_t(x) \tag{2.5}$$

be the "area", or the $(n-1)$-volume of Σ_t, and let V_t be the n-volume inclosed by Σ_t, $-\delta < t < \delta$. We recall the following variation formulas (computations given in the appendix A):

$$\frac{d}{dt}d\sigma_t(x) = H_t(x)\eta(x,t)d\sigma_t(x) \tag{2.6}$$

$$\frac{d}{dt}H_t(x) = -\Delta_{\Sigma_t}\eta(x,t) - \big(||\mathrm{II}_t||^2 + \mathrm{Ric}_{g_t}\nu(x,t),\nu(x,t)\big)\eta(x,t) =: L_{\Sigma_t}\eta(x,t) \tag{2.7}$$

$$\frac{d}{dt}A_t = \int_{\Sigma_t}\frac{d}{dt}\,d\sigma_t(x) = \int_{\Sigma_t} H_t(x)\eta(x,t)\,d\sigma_t(x) \tag{2.8}$$

$$\begin{aligned}\frac{d^2}{dt^2}A_t &= \int_{\Sigma_t}\left(\frac{d}{dt}H_t(x)\right)\eta(x,t)\,d\sigma_t(x) + H_t(x)\left(\frac{d}{dt}\eta(x,t)\right)\,d\sigma_t(x) + H_t(x)\eta(x,t)\frac{d}{dt}\,d\sigma_t(x)\\ &= \int_{\Sigma_t}\eta(x,t)L_{\Sigma_t}\eta(x,t)\,d\sigma_t(x) + H_t(x)\left(\frac{d}{dt}\eta(x,t)\right)\,d\sigma_t(x) + H_t^2(x)\eta(x,t)^2\,d\sigma_t(x)\end{aligned} \tag{2.9}$$

$$\frac{d}{dt}V_t = \int_{\Sigma_t}\eta(x,t)\,d\sigma_t(x) \tag{2.10}$$

$$\frac{d^2}{dt^2}V_t = \int_{\Sigma_t}\left(\frac{d}{dt}\eta(x,t)\right)\,d\sigma_t(x) + H_t(x)\eta(x,t)^2\,d\sigma_t(x) \tag{2.11}$$

From equation 2.8, we have

Theorem 2.1. *Σ is a critical point of the area functional A_t with respect to all variations η if and only if $H_\Sigma \equiv 0$, i.e., Σ is a minimal hypersurface.*

With a slight abuse of terminology, minimal hypersurfaces need not minimize area even locally. From calculus we know that we still need to check the second derivative. Now if Σ is a minimal hypersurface, then equation 2.8 implies that through a normal variation $\eta(x,t)\nu(x,t)$:

$$\frac{d^2}{dt^2}\Big|_{t=0}A_t = \int_\Sigma \eta(x)L_\Sigma\eta(x)\,d\sigma(x). \tag{2.12}$$

Therefore if $\int_\Sigma \eta(x)L_\Sigma\eta(x)\,d\sigma(x) \geq 0$ for all variations η, then Σ locally minimizes area A among all nearby hypersurfaces. This observation gives rise to the following definition:

Definition 2.1. *A minimal hypersurface Σ is called* stable *if*

$$\int_\Sigma \eta(x)L_\Sigma\eta(x)\,d\sigma(x) \geq 0 \textit{ for all variations } \eta. \tag{2.13}$$

A non-zero *CMC hypersurface Σ is called* stable *if*

$$\int_\Sigma \eta(x)L_\Sigma\eta(x)\,d\sigma(x) \geq 0,\ \textit{for all variations } \eta \textit{ with } \int_\Sigma \eta(x)\,d\sigma(x) = 0. \tag{2.14}$$

We thus call L_Σ the ***stability operator****.*

We have shown:

Theorem 2.2. *A stable minimal hypersurface locally minimizes area among all nearby hypersurfaces.*

Now a natural question is: do stable non-zero CMC hypersurfaces locally minimize area as well? It turns out this is true for volume-preserving variations:

Theorem 2.3. *Suppose Σ is a stable non-zero CMC hypersurface. Consider all variations η such that $\int_\Sigma \eta(x)dV(x) = 0$, and all Σ_t enclose the same n-volume, i.e., a volume preserving normal variation. Then Σ minimizes area among all such variational hypersurfaces.*

Proof. First note that if Σ is a non-zero CMC hypersurface, then Σ is a critical point of the area functional with respect to all variations with $\int_\Sigma \eta(x)d\sigma(x) = 0$. To see this, we use equation 2.8:

$$\frac{d}{dt}\Big|_{t=0} A_t = \int_\Sigma H(x)\eta(x)\, d\sigma(x) = H \int_\Sigma \eta(x)\, d\sigma(x) = 0. \tag{2.15}$$

Since the variation is also volume preserving, by equation 2.11 we have:

$$0 = \frac{d^2}{dt^2} V_t = \int_\Sigma \left(\frac{d}{dt}\eta(x,t)\right)\, d\sigma_t(x) + H_t(x)\eta(x,t)^2\, d\sigma_t(x) \tag{2.16}$$

since $\eta(x)$ has zero average on Σ, and H is constant. Thus the second variation of area (equation 2.9) becomes:

$$\begin{aligned}
\frac{d^2}{dt^2}\Big|_{t=0} A_t &= \int_\Sigma \eta(x)L_\Sigma\eta(x)\, d\sigma(x) + H\left(\frac{d}{dt}\Big|_{t=0}\eta(x,t)\right)\, d\sigma(x) + H^2\eta(x)^2\, d\sigma(x) \\
&= \int_\Sigma \eta(x)L_\Sigma\eta(x)\, d\sigma(x) + H\frac{d^2}{dt^2}V_t && \text{(Since } H \text{ is constant)} \\
&= \int_\Sigma \eta(x)L_\Sigma\eta(x)\, d\sigma(x) && (\eta \text{ is volume preserving}) \\
&\geq 0 && \text{(by stability of } \Sigma)
\end{aligned}$$

Therefore Σ locally minimizes area among nearby Σ_t hypersurfaces described as above. □

3 Examples of CMC hypersurfaces in Asymptotically Flat Manifolds

We are interested in finding stable CMC hypersurfaces. It turns out that stability of hypersurfaces is related to the eigenvalues of the stability operator. Suppose λ is an eigenvalue of L_Σ with eigenfunction η, then

$$\int_\Sigma \eta L_\Sigma \eta \, dV = \lambda \int_\Sigma \eta^2 \, dV. \tag{3.1}$$

Thus if the smallest eigenvalue of L_σ is non-negative, then the above is non-negative as well. Let's see some examples.

Example 3.1 (Sphere revisited)**.** *Let (M, g) be Euclidean space $(\mathbb{R}^n, \delta)$, and the hypersurface Σ be the standard round n sphere $S^{n-1}_R(0)$ of radius R centered at 0. The stability operator is then*

$$L_{S^{n-1}_R} = -\Delta_{S^{n-1}_R} - \frac{n-1}{R^2}. \tag{3.2}$$

Notice that $L_{S^{n-1}}$ and $-\Delta_{S^{n-1}_R}$ have the same set of eigenfunctions. Constant functions are eigenfunctions of $L_{S^{n-1}_R}$ with eigenvalues $\lambda_0 = -\frac{n-1}{R^2}$. $\lambda_1 = 0$ is an eigenvalue with eigenspace spanned by the coordinate functions $\{\frac{x^1}{R}, \frac{x^2}{R}, \cdots, \frac{x^n}{R}\}$. They corresponds to translational variations. In particular, S^{n-1}_R is not stable: translation of S^{n-1}_R will enclose the same volume.

Example 3.2 (Sphere in Schwarzschild)**.** *We now consider a 3-dimensional ambient manifold: $\left(\mathbb{R}^3 \backslash B_{\frac{m}{2}}, g = \left(1 + \frac{m}{2|x|}\right)^4 \delta\right)$, called the* Schwarzschild manifold *(see figure 5), where $m > 0$ is a constant and $B_{\frac{m}{2}}$ is the open ball with radius $\frac{m}{2}$. Thus the Schwarzschild manifold is a one-parameter family of manifolds.*

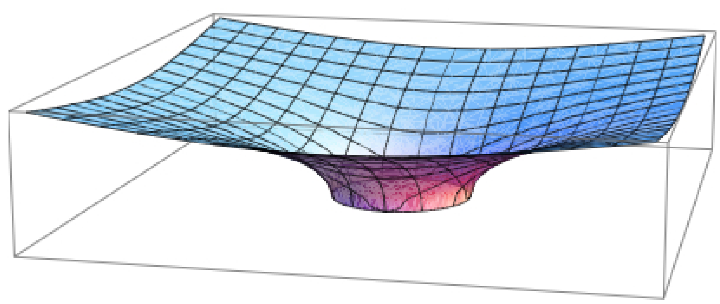

Figure 5: Schwarzschild manifold with one dimension suppressed. The boundary curve is the sphere of radius $\frac{m}{2}$. Picture courtesy of Jeffrey L. Jauregui, University of Pennsylvania

The Schwarzschild metric is conformal to the Euclidean metric, thus intuitively, as $|x| \to \infty$, the metric becomes flat. This is a special case of what is called an asymptotically flat *manifold, which we will discuss later. Now let S_R be the 2 sphere in Schwarzschild with radius R and the induced metric. By the conformal transformation of mean curvature formula, the mean curvature of S_R is:*

$$H_R = \left(2 - \frac{m}{R}\right) \frac{u(R)^{-3}}{R}, \tag{3.3}$$

where $u(|x|) := 1 + \frac{m}{2R}$. Therefore each such sphere S_R is a CMC surface. In particular, $S_{\frac{m}{2}}$ is a minimal surface. Notice that $\lim\limits_{R\to\infty} H_R = 0$, there must exist a local maximum in between $S_{\frac{m}{2}}$ and S_∞. Calculation shows that the sphere $S_{\frac{(2+\sqrt{3})m}{2}}$ has the largest mean curvature. Moreover, H_R is increasing for $\frac{m}{2} \leq R \leq \frac{(2+\sqrt{3})m}{2}$, and H_R is decreasing for $R \geq \frac{(2+\sqrt{3})m}{2}$ (see figure 6).

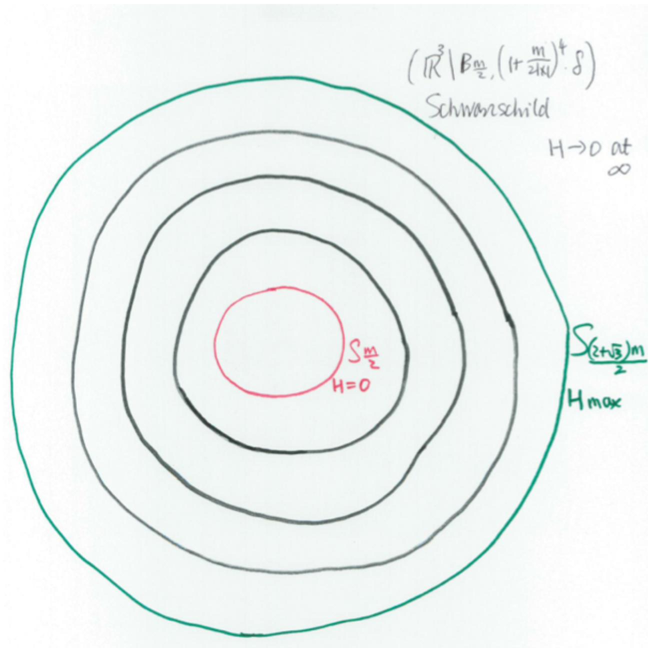

Figure 6: Foliations of CMC Spheres in Schwarzschild manifold

The question is: are those spheres stable? To answer this, we compute the stability operator for S_R as

$$L_{S_R} = -u^{-4}R^{-2}\Delta_{S^2} + \frac{-4R^2 + 8Rm - m^2}{2R^4u^6}, \tag{3.4}$$

where Δ_{S^2} is the Laplacian of the standard round unit sphere. The constant functions are eigenfunctions of L_{S_R} with eigenvalue $\lambda_0 = \frac{-4R^2+8Rm-m^2}{2R^4u^6}$, but those do not have zero average on S_R. The next smallest eigenvalue has multiplicity three: $\lambda_1 = \lambda_2 = \lambda_3 = \frac{6m}{R^3u^6}$ such that the eigenspace is spanned by coordinate functions $\{x^1, x^2, x^3\}$, which have zero average on Σ. Therefore S_R is stable for all $R \geq \frac{m}{2}$, and they foliate the entire Schwarzschild manifold, as indicated by Figure 6.

Both $\mathbb{R}^n$ and the Schwarzschild manifold are examples of an *asymptotically flat* manifolds:

Definition 3.1. *A complete Riemannian manifold (M^n, g) of dimension n is* asymptotically flat *if there exists a compact subset $K \subset M$ and a diffeomorphism*

$$\Phi : M^n \setminus K \longrightarrow \mathbb{R}^n \setminus \overline{B_1}, \tag{3.5}$$

which gives rise to a natural coordinate chart for $M\backslash K$. Moreover, we require that in this coordinate chart, the metric g and the scalar curvature R satisfies the following decay conditions:

1. $g_{ij}(x) = \delta_{ij} + O(|x|^{-p})$;
2. $|x||g_{ij,k}(x)| + |x|^2|g_{ij,kl}(x)| = O(|x|^{-p})$;
3. $|R(x) = O(|x|^{-q})$,

for all i, j, k, l and x, where $p > \frac{n-2}{2}$, and $q > n$.

An important question is: does there exists a foliation of stable CMC surfaces in asymptotically flat manifolds, just as in Schwarzschild? The existence of such foliation would allow us to define natural geometric coordinate systems on the manifold, much like the polar coordinate system. Rugang Ye [5] and Lan-Hsuan Huang [3] used a perturbation argument and proved such existence with some additional assumption (e.g., *strongly asymptotically flatness* and *Regge-Teitelboin condition*). The idea is that an asymptotically flat manifold approaches the flat Euclidean space at infinity, thus the spheres in such manifolds should approach to spheres in Euclidean space which are CMC. Therefore for a large sphere near infinity, one might be able to find a CMC surface *nearby*.

A Variation Formulas

A.1 First Variation of Area

Let Σ^{n-1} be an embedded closed (compact without boundary) hypersurface in a Riemannian manifold $(M^n, g, \overline{\nabla})$. Endow Σ with the induced metric. We consider a variation of Σ as follows:

$$F : \Sigma \times (-\delta, \delta) \longrightarrow M, \quad \delta > 0, \tag{A.1}$$

such that for all $x \in \Sigma_t := F(\Sigma, t)$, and $t \in (-\delta, \delta)$,

$$\frac{\partial}{\partial t} F(x, t) = \eta(x, t)\nu(x, t), \tag{A.2}$$

where η is a smooth function $\eta \in C^\infty(\Sigma \times (-\delta, \delta))$, and $\nu(x, t)$ is the unit outward normal vector to Σ_t at (x, t). Let g_t be the induced metric on Σ_t, and let ∇^t be the associated Levi-Civita connection. Let $d\sigma_t$ be the corresponding $(n-1)$-volume form on Σ_t, and A_t the $(n-1)$-volume. Let V_t be the n-volume enclosed by Σ_t. We shall refer to A_t as the *area* of Σ_t, and V_t the *volume*, in analogy to the case where Σ_t are surfaces in a 3-dimensional manifold. Let II_t and $H_t := \mathrm{tr}_{g_t}\mathrm{II}$ be the second fundamental form and the mean curvature of Σ with respect to $\nu(x, t)$ respectively. We first compute the variation of $d\sigma_t$. Let $\{U; x_1, x_2, \cdots, x_{n-1}\}$ be a local coordinate chart of Σ, then Σ_t can be locally parametrized as $\{x_1, x_2, \cdots, x_{n-1}, t\}$ with each fixed t. Let $g_t = (g_t)_{ij} dx^i dx^j$ be the local representation of the metric on Σ_t, $i, j = 1, 2, \cdots, n-1$.

$$
\begin{aligned}
\frac{\partial}{\partial t} d\sigma_t &= \frac{\partial}{\partial t}\sqrt{\det(g_t)} dx^1 \wedge dx^2 \cdots \wedge dx^{n-1} \\
&= \frac{1}{2}\frac{1}{\sqrt{\det(g_t)}}\det(g_t)\cdot \mathrm{trace}\left(g_t^{-1}\frac{\partial}{\partial t}g_t\right) dx^1 \wedge dx^2 \wedge \cdots \wedge dx^{n-1} \qquad \text{(A.3)} \\
&= \frac{1}{2}\sqrt{\det(g_t)}\mathrm{trace}\left(g_t^{-1}\frac{\partial}{\partial t}g_t\right) dx^1 \wedge dx^2 \wedge \cdots \wedge dx^{n-1}
\end{aligned}
$$

where equation $(A.3)$ follows from the identity

$$
\frac{d}{dt}\det(A) = \det(A)\mathrm{trace}\left(A^{-1}\frac{d}{dt}A\right), \qquad \text{(A.4)}
$$

for any square matrix A with entries functions of t. Now

$$
\begin{aligned}
\frac{\partial}{\partial t}(g_t)_{ij} &= \frac{\partial}{\partial t}\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\rangle = \langle \overline{\nabla}_{\frac{\partial}{\partial t}}\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\rangle + \langle \frac{\partial}{\partial x^i}, \overline{\nabla}_{\frac{\partial}{\partial t}}\frac{\partial}{\partial x^j}\rangle \\
&= \langle \overline{\nabla}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial t}, \frac{\partial}{\partial x^j}\rangle + \langle \frac{\partial}{\partial x^i}, \overline{\nabla}_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial t}\rangle \qquad (\overline{\nabla} \text{ is torsion free}) \\
&= \langle \overline{\nabla}_{\frac{\partial}{\partial x^i}}\eta\nu, \frac{\partial}{\partial x^j}\rangle + \langle \frac{\partial}{\partial x^i}, \overline{\nabla}_{\frac{\partial}{\partial x^j}}\eta\nu\rangle \\
&= 2\eta \mathrm{II}_t\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right). \qquad \text{(A.5)}
\end{aligned}
$$

Now plug (A.5) into (A.3):

$$
\frac{\partial}{\partial t}d\sigma_t = \frac{1}{2}\sqrt{\det(g_t)}\mathrm{trace}\left(g_t^{-1}\cdot 2\eta\cdot \mathrm{II}_t(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})\right) dx^1 \wedge dx^2 \wedge \cdots \wedge dx^{n-1} = H_t \eta d\sigma_t \qquad \text{(A.6)}
$$

Therefore

$$
\frac{\partial}{\partial t}A_t = \int_{\Sigma_t}\frac{\partial}{\partial t}d\sigma_t = \int_{\Sigma_t} H_t(x)\eta(x,t)\, d\sigma_t(x). \qquad \text{(A.7)}
$$

A.2 Second Variation of Area

Now we compute the first variation of mean curvature H_t, which gives rise to the second derivative of area. Recall that $H_t = g_t^{ij}(\mathrm{II}_t)_{ij}$ in local coordinates, $i, j = 1, 2, \cdots, n-1$. Thus

$$
\frac{\partial}{\partial t}H_t = \frac{\partial}{\partial t}g_t^{ij}(\mathrm{II}_t)_{ij} + g_t^{ij}\frac{\partial}{\partial t}(\mathrm{II}_t)_{ij} \qquad \text{(A.8)}
$$

Since $0 = \frac{\partial}{\partial t}(g_t g_t^{-1}) = (\frac{\partial}{\partial t}g_t)g_t^{-1} + g_t(\frac{\partial}{\partial t}g_t^{-1})$, we have $\frac{\partial}{\partial t}g_t^{-1} = -g_t^{-1}(\frac{\partial}{\partial t}g_t)g_t^{-1}$. Thus the first term in the above becomes

$$
\frac{\partial}{\partial t}g_t^{ij}(\mathrm{II}_t)_{ij} = -g_t^{ik}\left(\frac{\partial}{\partial t}(g_t)_{kl}\right)g_t^{lj}(\mathrm{II}_t)_{ij} = -g_t^{ik}2\eta(\mathrm{II}_t)_{kl}g_t^{lj}(\mathrm{II}_t)_{ij} = -2\eta||\mathrm{II}_t||^2 \qquad \text{(A.9)}
$$

We now compute the derivative of the second fundamental form.

$$\begin{aligned}
\frac{\partial}{\partial t}(\mathrm{II}_t)_{ij} &= \frac{\partial}{\partial t}\langle \overline{\nabla}_{\frac{\partial}{\partial x^i}}\nu, \frac{\partial}{\partial x^j}\rangle \\
&= \langle \overline{\nabla}_{\frac{\partial}{\partial t}}\overline{\nabla}_{\frac{\partial}{\partial x^i}}\nu, \frac{\partial}{\partial x^j}\rangle + \langle \overline{\nabla}_{\frac{\partial}{\partial x^i}}\nu, \overline{\nabla}_{\frac{\partial}{\partial t}}\frac{\partial}{\partial x^j}\rangle \\
&= \langle \overline{\nabla}_{\frac{\partial}{\partial x^i}}\overline{\nabla}_{\frac{\partial}{\partial t}}\nu, \frac{\partial}{\partial x^j}\rangle + \langle (\overline{\nabla}_{\frac{\partial}{\partial t}}\overline{\nabla}_{\frac{\partial}{\partial x^i}} - \overline{\nabla}_{\frac{\partial}{\partial x^i}}\overline{\nabla}_{\frac{\partial}{\partial t}} - \overline{\nabla}_{[\frac{\partial}{\partial x^i},\frac{\partial}{\partial t}]})\nu, \frac{\partial}{\partial x^j}\rangle + \langle \overline{\nabla}_{\frac{\partial}{\partial x^i}}\nu, \overline{\nabla}_{\frac{\partial}{\partial t}}\frac{\partial}{\partial x^j}\rangle \\
&= \langle \overline{\nabla}_{\frac{\partial}{\partial x^i}}\overline{\nabla}_{\frac{\partial}{\partial t}}\nu, \frac{\partial}{\partial x^j}\rangle + \langle R_g(\frac{\partial}{\partial t}, \frac{\partial}{\partial x^i})\nu, \frac{\partial}{\partial x^j}\rangle + \langle \overline{\nabla}_{\frac{\partial}{\partial x^i}}\nu, \overline{\nabla}_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial t}\rangle \\
&= \langle \overline{\nabla}_{\frac{\partial}{\partial x^i}}\left(-\nabla_{\Sigma_t}\eta\right), \frac{\partial}{\partial x^j}\rangle + \eta\langle R(\nu, \frac{\partial}{\partial x^i})\nu, \frac{\partial}{\partial x^j}\rangle + \eta\langle \overline{\nabla}_{\frac{\partial}{\partial x^i}}\nu, \overline{\nabla}_{\frac{\partial}{\partial x^j}}\nu\rangle
\end{aligned}$$

where we have used two lemmas, which will be proved below:

Lemma 1. *$\overline{\nabla}_{\frac{\partial}{\partial x^i}}\nu$ is tangential, i.e., $\langle \overline{\nabla}_{\frac{\partial}{\partial x^i}}\nu, \nu\rangle = 0$.*

Lemma 2. *$\overline{\nabla}_{\frac{\partial}{\partial t}}\nu = -\nabla_{\Sigma_t}\eta$, where ∇_{Σ_t} is the surface gradient on Σ_t.*

Therefore

$$\begin{aligned}
g_t^{ij}\frac{\partial}{\partial t}(\mathrm{II}_t)_{ij} &= -\Delta_{\Sigma_t}\eta - \eta g_t^{ij}\langle R(\nu, \frac{\partial}{\partial x^i}), \frac{\partial}{\partial x^j}, \nu\rangle + \eta||\mathrm{II}_t||^2 \\
&= -\Delta_{\Sigma_t}\eta - \eta g^{ij}\langle R(\nu, \frac{\partial}{\partial x^i}), \frac{\partial}{\partial x^j}, \nu\rangle + \eta||\mathrm{II}_t||^2 \quad \text{(ambient metric } g \text{ trace)} \\
&= -\Delta_{\Sigma_t}\eta - \eta\mathrm{Ric}_g(\nu,\nu) + \eta||\mathrm{II}_t||^2.
\end{aligned}$$

where we used

Lemma 3. *$g_t^{ij}\langle \overline{\nabla}_{\frac{\partial}{\partial x^i}}\left(\nabla_{\Sigma_t}\eta\right), \frac{\partial}{\partial x^j}\rangle = \Delta_{\Sigma_t}\eta$.*

and

Lemma 4. *$g_t^{ij}\langle \overline{\nabla}_{\frac{\partial}{\partial x^i}}\nu, \overline{\nabla}_{\frac{\partial}{\partial x^j}}\nu\rangle = ||II_t||^2$.*

Combining above, we get

$$\frac{\partial}{\partial t}H_t = -\Delta_{\Sigma_t}\eta - \eta\left(\mathrm{Ric}_g(\nu,\nu) + ||\mathrm{II}_t||^2\right) =: L_{\Sigma_t}\eta, \tag{A.10}$$

where L_{Σ_t} is called the *stability operator* of Σ_t. The second variation of area is then given by:

$$\frac{\partial}{\partial t}A_t = \int_\Sigma \eta(x,t)(L_{\Sigma_t}\eta)(x,t)\,d\sigma_t(x) + H_t(x)\left(\frac{\partial}{\partial t}\eta(x,t)\right)\,d\sigma_t(x) + H_t^2\eta(x,t)^2\,d\sigma_t(x). \tag{A.11}$$

Now we verity the above lemmas.

Proof of Lemma 1. Since ν is the unit outward normal vector field, we have

$$0 = \frac{\partial}{\partial x^i}\langle \nu, \nu\rangle = 2\langle \overline{\nabla}_{\frac{\partial}{\partial x^i}}\nu, \nu\rangle. \tag{A.12}$$

Thus $\overline{\nabla}_{\frac{\partial}{\partial x^i}}\nu$ is tangential. Similarly, $\overline{\nabla}_{\frac{\partial}{\partial t}}\nu$ is also tangential. $\square$

Proof of Lemma 2. Recall that the surface gradient is defined as

$$\nabla_{\Sigma_t}\nu := \nabla\nu - \langle \nabla\nu, \nu\rangle\nu, \tag{A.13}$$

that is, the tangential component of the gradient with respect to the ambient metric. For any point $p \in \Sigma_t$, choose geodesic normal coordinates $\{U; e_1, e_2, \cdots, e_n\}$ around p such that $e_1, e_2, \cdots, e_{n-1}$ span $T_p\Sigma_t$, and $e_n = \nu$. Since $\overline{\nabla}_{e_i}\nu$ is tangential, it suffices to show that $\langle \overline{\nabla}_{\frac{\partial}{\partial t}}\nu, e_i\rangle(p) = \langle -\nabla_{\Sigma_t}\eta, e_i\rangle(p)$, for $i = 1, 2, \cdots, n-1$. Indeed:

$$\begin{aligned}
\langle \overline{\nabla}_{\frac{\partial}{\partial t}}\nu, e_i\rangle(p) &= -\langle \nu, \overline{\nabla}_{\frac{\partial}{\partial t}}e_i\rangle(p) = -\langle \nu, \overline{\nabla}_{e_i}\frac{\partial}{\partial t}\rangle(p) && (\overline{\nabla}\text{ is torsion free})\\
&= -e_i(\eta)\langle \nu, \nu\rangle(p) - \eta\langle \nu, \overline{\nabla}_{e_i}\nu\rangle(p)\\
&= -e_i(\eta)(p) && (\overline{\nabla}_{e_i}\nu\text{ is tangential})\\
&= -\langle \sum_{j=1}^{n-1} e_j(\eta)e_j, e_i)\rangle(p)\\
&= \langle -\nabla_{\Sigma_t}\eta, e_i\rangle(p).
\end{aligned}$$

Since p is arbitrary, $\overline{\nabla}_{\frac{\partial}{\partial t}}\nu = -\nabla_{\Sigma_t}\eta$, as desired. $\square$

Proof of Lemma 3. First note that $g_t^{ij}\langle \overline{\nabla}_{\frac{\partial}{\partial x^i}}(\nabla_{\Sigma_t}\eta), \frac{\partial}{\partial x^j}\rangle = g_t^{ij}\langle \nabla^t_{\frac{\partial}{\partial x^i}}(\nabla_{\Sigma_t}\eta), \frac{\partial}{\partial x^j}\rangle$. But

$$g_t^{ij}\langle \nabla^t_{\frac{\partial}{\partial x^i}}(\nabla_{\Sigma_t}\eta), \frac{\partial}{\partial x^j}\rangle = g_t^{ij}\nabla^t_{\frac{\partial}{\partial x^j}}\nabla^t_{\frac{\partial}{\partial x^i}}(\nabla_{\Sigma_t}\eta) = \mathrm{div}_{g_t}(\nabla_{\Sigma_t}\eta) = \Delta_{\Sigma_t}\eta. \tag{A.14}$$

$\square$

Proof of Lemma 4. Define vector fields $X := \sum_{i=1}^{n-1}\overline{\nabla}_{\frac{\partial}{\partial x^i}}\nu$ and $Y := \sum_{j=1}^{n-1}\overline{\nabla}_{\frac{\partial}{\partial x^j}}\nu$. Using local coordinates, we can also write $X = \sum_{k=1}^{n-1}X^k\frac{\partial}{\partial x^k}$ and $Y = \sum_{l=1}^{n-1}Y^l\frac{\partial}{\partial x^l}$. Then

$$\begin{aligned}
||\mathrm{II}_t||^2 &= g_t^{ij}g_t^{kl}(\mathrm{II}_t)_{ik}(\mathrm{II}_t)_{jl} = g_t^{ij}g_t^{kl}\langle X, \frac{\partial}{\partial x^k}\rangle\langle Y, \frac{\partial}{\partial x^l}\rangle\\
&= g_t^{ij}g_t^{kl}X^\alpha(g_t)_{\alpha k}Y^\beta(g_t)_{\beta l}\\
&= g_t^{ij}(g_t)_{\alpha\beta}X^\alpha Y^\beta\\
&= g_t^{ij}\langle X, Y\rangle\\
&= g_t^{ij}\langle \overline{\nabla}_{\frac{\partial}{\partial x^i}}\nu, \overline{\nabla}_{\frac{\partial}{\partial x^j}}\nu\rangle
\end{aligned}$$

as desired. $\square$

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