Inverse Mean Curvature Vector Flows in Spacetimes*

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Contents

1 Introduction: 3 minutes

• Welcome to my defense presentation. The title of my talk today is: **Inverse Mean Curvature Vector Flows**. What I did in my thesis is that I showed that *there exist infinitely many examples of spacetimes that have smooth inverse mean curvature flow solutions that exist for all time*. Prior to our work, solutions to such flow were only known to exist in spherical symmetric and static spacetimes. **The technique used in our construction may be important to prove the Spacetime Penrose Conjecture**.

• This talk consists of four part: First, I will introduce some of the background material including inverse mean curvature vector flow and the Penrose conjecture; Second, I will discuss how inverse mean curvature vector flow always works in spherical symmetry; After that, I will get to my work on constructing non-spherically symmetric

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spacetimes that have inverse mean curvature vector flow solutions; Lastly, I will statement some open problems.

2 Motivation: 18 minutes

2.1 Riemannian Penrose Inequality

General relativity is the leading theory in describing the large scale structure of the universe. It has achieved many success including predictions of the big bang and black holes. However, the notion of mass of a given region, or *quasi-local mass*, as well as its relationships with the global mass of our spacetime, are still not very well understood. (Draw the following illustration.)

pointwise energy density $\leftarrow$ local mass of a region $\rightarrow$ global mass of the spacetime.

(Narrate: The pointwise energy density and the global mass of a spacetime are two relatively well understood quantities. However, the local mass that fits in the middle is still not well understood.)

Given a spacetime N that admits a totally geodesic (zero second fundamental form), asymptotically flat spacelike hypersurface M, the Riemannian Penrose conjecture states that:

Theorem 2.1 (Riemannian Penrose Inequality)**.** *Draw picture first and then narrate:*

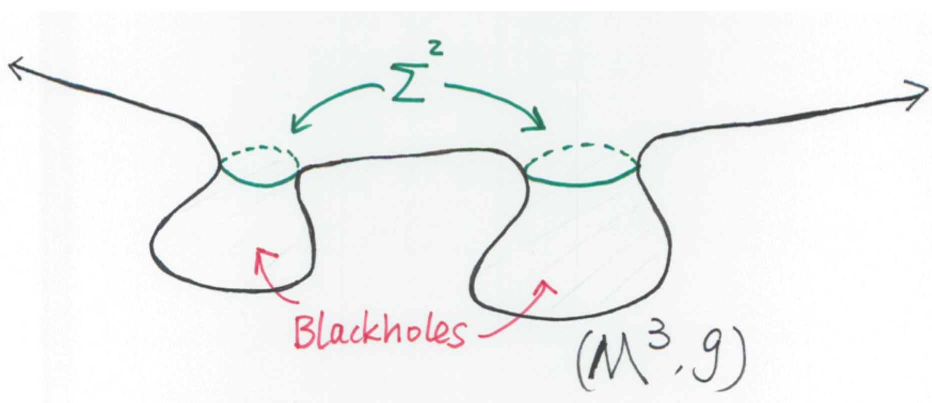

Figure 1: Totally geodesic spacelike slice with outermost minimal surface boundaries

Given a complete asymptotically flat Riemannian manifold (M^3, g) with nonnegative scalar curvature and a compact outermost minimal (i.e. not enclosed by a surface with equal of less area) surface Σ^2, then

$$m_{ADM} \geq \sqrt{\frac{|\Sigma|}{16\pi}}, \tag{2.1}$$

where m_{ADM} is the ADM mass of M (named after Arnowitt, Deser and Misner), *and $|\Sigma|$ is the area of Σ.*

Σ can be viewed as the apparent horizon of black holes. Intuitively, this inequality is simply saying that the total mass of the manifold M is bounded from below by the mass contributed by the black holes. This inequality was first proved by Huisken-Ilmanen using inverse mean curvature flow in the case of a single blackhole, and then by H. Bray for arbitrary number of blackholes using conformal flow of metrics. The zero blackhole case is also known as the **Positive Mass Theorem**, which was first proved by Schoen and Yau using minimal surface techiques, and later by Witten using spinors.

2.2 Inverse Mean Curvature Flow

Huisken and Ilmanen's approach was based on the Geroch, Jang/Wald monotonicity formula of Hawking mass under inverse mean curvature flow.

Definition 2.1. *Given a closed surface Σ in M. The inverse mean curvature flow of Σ is a smooth family of surfaces $F : \Sigma \times [0,T] \longrightarrow M$, such that*

$$\frac{\partial F}{\partial t} = \frac{\nu_t}{H_t}, \quad t \in [0,T], (x,t) \in \Sigma_t := F(\Sigma, t), \tag{2.2}$$

(Draw inverse mean curvature vector flow here after this equation.) *where ν_t is the outward unit normal to Σ_t, and H_t is the scalar mean curvature of Σ_t in M.*

Geroch, Jang/Wald discovered a connection between smooth inverse mean curvature flow and a quasi-local mass called Hawking mass. The Hawking mass of a closed surface Σ is defined to be:

$$m_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}} \left(1 - \frac{1}{16\pi}\int_\Sigma (H_\Sigma)^2 \, dA_\Sigma\right), \tag{2.3}$$

• If Σ is outermost minimal, then its Hawking mass equals to $\sqrt{\frac{|\Sigma|}{16\pi}}$. Notice that this is the right hand side of the Riemannian Penrose inequality (2.1).

• Geroch, Jang/Wald showed that if $\{\Sigma_t\}$ is a smooth solution to inverse mean curvature flow, then the Hawking mass monotonically non-decreasing (Here, simply write $m_H(\Sigma_t)$ increases).

• Let $S_r(0)$ be coordinate sphere of radius r in an asymptotically flat coordinate chart of M. Huisken and Ilmanen showed that:

$$\lim_{r\to\infty} m_H(S_r) = m_{ADM}(M).$$

• Combining the above, we have that if $\{\Sigma_t\}$ is a smooth solution to inverse mean curvature flow, and Σ_t approaches to large coordinate spheres near infinity sufficiently fast, then:

$$m_{ADM}(M) = \lim_{t\to\infty} m_H(\Sigma_t) \geq m_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}},$$

which proves the Riemannian Penrose conjecture.

• However, the inverse mean curvature flow is undefined when $H_\Sigma = 0$. **Thus, we don't even know how to start the flow if the initial surface is minimal**. Huisken and Ilmanen defined a weak notion of solutions to inverse mean curvature flow via a level set formulation. Roughly speaking, the flow surfaces will "jump" when the mean curvature is about to be zero. Huisken and Ilmanen showed is that **the Hawking mass satisfies all the key properties of the smooth case, in particular it is non-decreasing during jumps as well**. Using this, they proved the Riemannian Penrose conjecture in the case of a single black hole, i.e. connected outmost minimal boundary. This is a wonderful example of solving a differential geometric problem using analysis.

• Conclusion: the inverse mean curvature flow naturally bridges the local mass (Hawking mass) of a outermost minimal surface with the global mass (ADM) mass of a hypersurface in a spacetime: (Draw the following illstruation.)

Hawking mass of apparent horizon of BHs $\xrightarrow{IMCF}$ ADM mass of hypersurface in spacetime.

2.3 Inverse Mean Curvature Vector Flow

• Up until this point assumes that the spacetime has a spacelike hypersurface with zero second fundamental form, that is, a totally geodesic a slice. However, a spacetime generally has no such totally geodesic hypersurface, since the second fundamental form has six components, and a hypersurface only has one degree of freedom. Thus, it is desirable to show that the total mass of the spacetime is bounded by the mass contributed by the blackholes *without* this assumption. This is the **Spacetime Penrose Conjecture**, which is still open today.

A **possible candidate** for proving this conjecture is the codimension-two analogue of inverse mean curvature flow, called the *inverse mean curvature vector flow*. (Write down the IMCVF equation here:)

Definition 2.2. *Given a closed surface Σ in a spacetime N. An inverse mean curvature vector flow of Σ is a smooth family of surfaces $F : \Sigma \times [0,T] \longrightarrow N$ such that:*

$$\frac{\partial F}{\partial s} = \vec{I}_s := -\frac{\vec{H}_s}{\langle \vec{H}_s, \vec{H}_s\rangle}. \tag{2.4}$$

where $\vec{H}_s$ is the mean curvature vector of $\Sigma_s := F(\Sigma, s)$. $\vec{I}_s$ defined above is called the inverse mean curvature vector.

There are two problems with this flow:

(1) First, **inverse mean curvature vector flow might not have solutions given any initial surface**. As Huisken and Ilmanen pointed out, the inverse mean curvature

vector flow equation (2.4) is a 2×2 forward-backward parabolic system of equations. Forward parabolic equations are nice and smooths things out, but the problem is that **backward parabolic equations do not have a general existence theory**. For instance, the reverse heat flow equation is backward parabolic, and singularities usually develop instantaneously given certain initial conditions. However, the reverse heat flow exists for time $t > 0$ if we first perform the heat flow, which is forward parabolic, for time t and then flow backwards.

(2) Second, **for some initial surface, even the inverse mean curvature vector flow exist, the Hawking mass of the initial surface won't give us a lower bound on the total mass of the spacetime simply because the former is too large**. To illustrate this, take a $t =$ constant slice in the Minkowski spacetime. The round sphere in that slice has zero Hawking mass (Prepare for question here: why the Hawking mass is zero?). Spacial perturbations will decrease the Hawking mass making it negative, whereas timelike perturbations will increase the Hawking mass making it positive. During inverse mean curvature vector flow, the spacial "wiggles" will smooth out due to the parabolic nature of the flow. However, timelike "wiggles" will get amplified since the flow is reverse parabolic in the timelike directions. With these surfaces with positive Hawking mass, inverse mean curvature vector flow will not provide a lower bound for the ADM mass of Minkowski space (ADM mass of a spacetime is also defined), which is zero.

• However, these two problems seem to solve each other because they are both suggesting that inverse mean curvature vector flow exists only when the "right" initial surface is given. On the other hand, inverse mean curvature vector flow **always** works in spherically symmetric spacetimes. It is thus important to understand the geometry of spherical symmetry.

3 Spherically Symmetric Spacetimes: 10 minutes

Roughly speaking, a spacetime is said to be *spherically symmetric* if its isometry group contains a subgroup that is isomorphic to the rotation group $SO(3)$. Given a spherically symmetric spacetime $(N^4, \overline{g})$ satisfying the dominant energy condition that admits a coordinate chart $\{t, r, \theta, \phi\}$, such that $\overline{g}$ takes the form: (Keep this coordinate representation.)

$$g = \begin{array}{c} \\ t \\ r \\ \theta \\ \phi \end{array} \begin{array}{c} \begin{array}{cccc} t & r & \theta & \phi \end{array} \\ \begin{pmatrix} -v^2(t,r) & 0 & 0 & 0 \\ 0 & u^2(t,r) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \end{array} \tag{3.1}$$

where u and v are smooth functions of t and r only. Roughly speaking, every spherically symmetric spacetimes outside blackholes can be expressed this way. For instance, the Minkowski spacetime $(\mathbb{R}^{3,1}, -dt^2 + dr^2 + r^2 d\Omega^2)$ and the Schwarzschild spacetime $(\mathbb{R}^4 \backslash \{0\}, -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2)$.

• Upshot: Within each $t =$ constant slice, smooth inverse mean curvature vector flow of coordinate sphere $S_{t,r}$ exists for all time. To see this, we can compute that:

$$\vec{H}_{t,r} = -\frac{2}{r}\frac{1}{u^2}\partial_r. \tag{3.2}$$

So the inverse mean curvature vector is:

$$\vec{I}_{t,r} = -\frac{\vec{H}_{t,r}}{\langle \vec{H}_{t,r}, \vec{H}_{t,r} \rangle} = \frac{\frac{2}{r}\frac{1}{u^2}\partial_r}{\langle -\frac{2}{r}\frac{1}{u^2}\partial_r, -\frac{2}{r}\frac{1}{u^2}\partial_r \rangle} = \frac{\frac{2}{r}\frac{1}{u^2}\partial_r}{\frac{4}{r^2}\frac{1}{u^4}u^2} = \frac{r}{2}\partial_r. \tag{3.3}$$

Therefore, inverse mean curvature vector flow of $S_{t,r}$ is a reparametrization of radial flow, and hence is smooth and exits for all time. Moreover, the Hawking mass is monotone **if the spacetime satisfies the dominant energy condition**. Recall that:

$$m_H(S_{t,r}) = \sqrt{\frac{|S_{t,r}|}{16\pi}} \left(1 - \frac{1}{16\pi}\int_{S_{t,r}} \overline{g}(\vec{H}_{t,r}, \vec{H}_{t,r})\, dA_{t,r}\right), \tag{3.4}$$

where $dA_{t,r}$ is the area form of $S_{t,r}$, and $|S_{t,r}|$ is the area of $S_{t,r}$. Note that (Simply state this during the presentation by the side) $|S_{t,r}| = 4\pi r^2$ since the metric on $S_{t,r}$ is the standard round metric on two-spheres with radius r.

Plug the mean curvature vector of $S_{t,r}$ (3.2) into the Hawking mass equation (3.4), we get:

$$m_H(S_{t,r}) = \sqrt{\frac{4\pi r^2}{16\pi}} \left(1 - \frac{1}{16\pi}\int_{S_{t,r}} \overline{g}\left(-\frac{2}{r}\frac{1}{u^2}\partial_r, -\frac{2}{r}\frac{1}{u^2}\partial_r\right) dA_{t,r}\right)$$

(This line follows immediately from Equation (3.4) above)

$$= \frac{r}{2}\left(1 - \frac{1}{16\pi}\int_{S_{t,r}} \frac{4}{r^2}\frac{1}{u^4}\overline{g}(\partial_r, \partial_r)\, dA_{t,r}\right) \quad \text{(Skip this step)}$$

$$= \frac{r}{2}\left(1 - \frac{1}{16\pi}\int_0^{2\pi}\int_0^{\pi} \frac{4}{r^2}\frac{1}{u^2} r^2 sin\theta\, d\theta\, d\phi\right) \quad \text{(Skip this step)}$$

$$= \frac{r}{2}\left(1 - \frac{1}{4\pi}\frac{1}{u^2}4\pi\right) \quad \text{(Skip this step)}$$

$$= \frac{r}{2}\left(1 - \frac{1}{u^2}\right). \quad \text{(Discuss the Hawking mass of sphere in Minkowski space)}$$

By Equation (3.3), the variation of the Hawking mass of $S_{t,r}$ along inverse mean curvature vector flow is given by:

$$\vec{I}_{t,r}(m_H(S_{t,r})) = \frac{r}{2}\frac{\partial m_H(S_{t,r})}{\partial_r} = \frac{r}{2}\left[\frac{1}{2}\left(1 - \frac{1}{u^2}\right) + r\frac{u_{,r}}{u^3}\right] = \frac{r^3}{4v^2}G(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) \geq 0.$$

where G is the Einstein curvature tensor of $(N^4, \overline{g})$, and the last inequality follows from the dominant energy condition. All the detailed computations can be found in Proposition 2.2 in my thesis.

• Conclusion: A spherically symmetric spacetime $(N^4, \overline{g})$ that admit such a coordinate chart can then be foliated by $t =$ constant spacelike hyperplanes, and each hyperplane can be foliated by smooth inverse mean curvature vector flow spheres (see Figure 2). (Draw this slice bigger, and only draw three spheres with large space between them.)

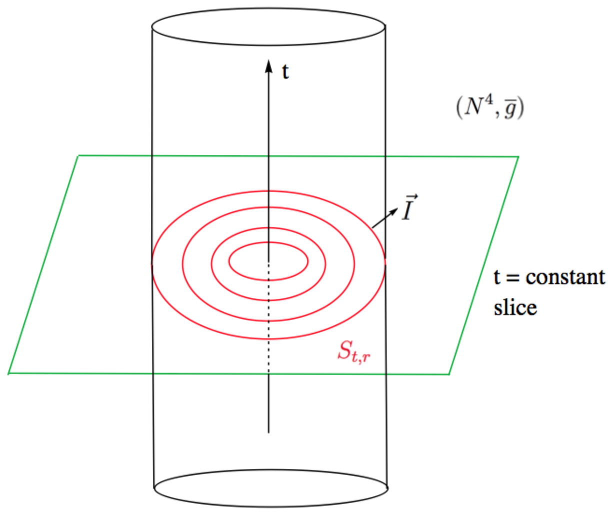

Figure 2: IMCVF of coordinate spheres $S_{t,r}$ in spherically symmetric spacetime (3.1).

• Thesis Problem Statement: **In spherically symmetric spacetimes, the "right " initial surfaces are spherically symmetric spheres. Had we chosen any other spheres, inverse mean curvature vector flow would not exist**. Is this situation **generic** in the sense that given any spacetime we can find the "right" initial surface? (Write the next sentence down) Can we find *non-spherically symmetric* spacetimes that admit smooth solutions to inverse mean curvature vector flow as well?

4 Now here is what I did: Spacetimes that Admit Inverse Mean Curvature Vector Flow Coordinate Chart: 22 minutes

Sine we want to generalize the spherically symmetric case, in particular the special double foliation, let's first assume that a spacetime admits such special double foliation, and see what kind of topological and geometric constraints this double foliation puts on the spacetime.

Suppose a spacetime $(N^4, \overline{g})$ admits the following special foliation: N is foliated by spacelike hyperplanes, and then within each hyperplane, smooth IMCVF of spheres exists and foliates the entire hyperplane. Consequently, the mean curvature vector of the flow spheres are tangential to the hyperplane. **Notice that this foliation puts restrictions on the topology of** N, as N has to be topologically equivalent to $(\mathbb{R}^3 \backslash B_1) \times \mathbb{R}$, where B_1 is the closed unit ball in $\mathbb{R}^3$.

Suppose $(N^4, \overline{g})$ admits such a special foliation. We can use this to define coordinates that generalize the spherically symmetric coordinates. (Talk while you draw Figure 3.)

- We define the t-coordinate by setting each hyperplane as $t =$ constant, thus the t-coordinate tells us which hyperplane we are on.
- For each IMCVF sphere, define $A = 4\pi r^2$, where A is the area of that sphere. This defines a very natural r-coordinate. Since IMCVF is area expanding, the r-coordinate is well-defined. **For simplicity we assume that** $r \geq 1$**, i.e. the initial spheres on each hyperplane have area** 4π.
- Then define (θ, ϕ)-coordinates on an initial sphere, $0 < \theta < \pi$ and $0 < \phi < 2\pi$, such that the area form satisfies

$$dA_0 = \frac{A(0)}{4\pi} \sin\theta d\theta d\phi = \sin\theta d\theta d\phi, \tag{4.1}$$

where $A(0)$ is the area of the initial sphere. Extend (θ, ϕ) by setting them to be constant along perpendicular directions of the initial sphere, such that (θ, ϕ) coordinates are defined for each sphere. By the extension, $\frac{\partial}{\partial r}$ will be perpendicular to each sphere. It can be shown, using the first variation of area, that the equation for the area form (4.1) will be preserved: $dA_r = \frac{A(r)}{4\pi} \sin\theta d\theta d\phi$ (Skip this in the talk. Explain if someone asks.)*. (At this point, Figure 3 should be drawn.)

- <u>Conclusion</u>: If a spacetime $(N^4, \overline{g})$ admits the above special double foliation, (Write this down and use $\Rightarrow$. Save head room for "Theorem" once the converse if proved.) then there exists a coordinate chart $\{t, r, \theta, \phi\}$ of N, such that in this coordinate chart the metric has the form (Do not erase):

$$\overline{g} = \begin{array}{c} \\ t \\ r \\ \theta \\ \phi \end{array} \begin{array}{c} \begin{array}{cccc} t & r & \theta & \phi \end{array} \\ \begin{pmatrix} -v^2 & d & e & f \\ d & u^2 & 0 & 0 \\ e & 0 & a & c \\ f & 0 & c & b \end{pmatrix} \end{array} \tag{4.2}$$

where u, v, a, b, c, d, e, f are smooth functions on N, and the following four conditions

*In smooth IMCVF, $\frac{d}{ds}(dA_s) = -\langle \vec{I}_s, \vec{H}_s \rangle dA_s = dA_s$, where dA_s is the area form of Σ_s. The solution to this equation is $dA_s = e^s dA_0$. The area of Σ_s is thus given by $A(s) = A(0)e^s = 4\pi r^2$, by the definition of the r-coordinate. Thus $e^s \frac{A(0)}{4\pi} = r^2$. Therefore the area form in the r parameter is $dA_r = e^s dA_0 = e^s \frac{A(0)}{4\pi} \sin\theta d\theta d\phi = r^2 \sin\theta d\theta d\phi = \frac{A(r)}{4\pi} \sin\theta d\theta d\phi$.

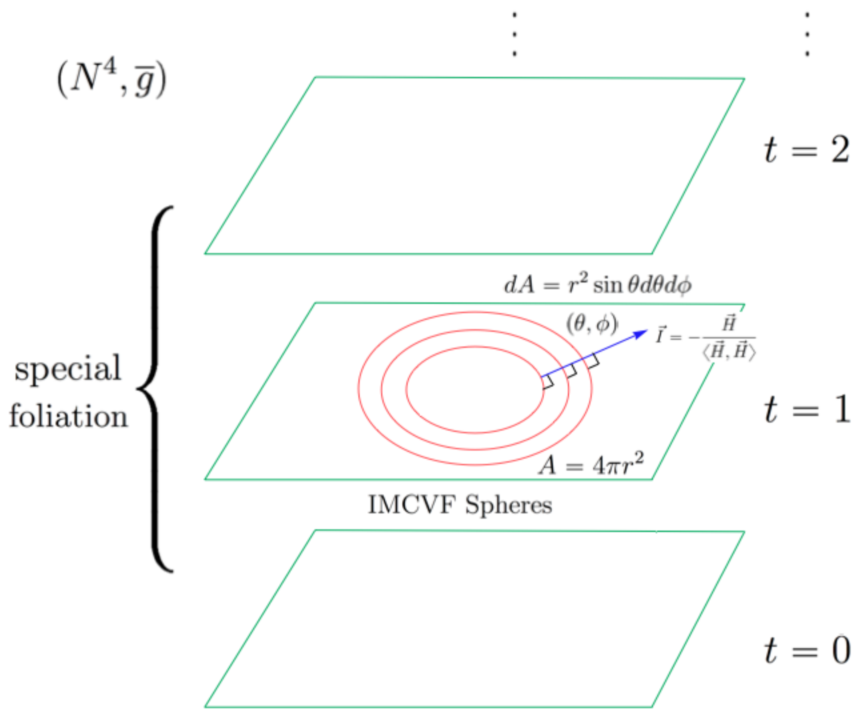

Figure 3: Special foliation of a spacetime $(N^4, \overline{g})$: first foliated by hyperplanes, then each hyperplane is foliated by IMCVF of spheres. This generalizes the spherically symmetric case in Figure 2.

are satisfied (Do not erase these four conditions):

$$(1) \quad \langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta} \rangle = 0; \tag{4.3}$$

$$(2) \quad \langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \phi} \rangle = 0; \tag{4.4}$$

$$(3) \quad dA_{t,r} = r^2 \sin\theta d\theta d\phi \text{ (i.e. } ab - c^2 = r^4 \sin^2\theta); \tag{4.5}$$

$$(4) \quad \vec{H}_{t,r} \text{ is tangential to the } t = \text{constant hyperplane;} \tag{4.6}$$

where $dA_{t,r}$ and $\vec{H}_{t,r}$ are the area form and the mean curvature vector of the coordinate sphere $S_{t,r}$, respectively.

• It can be shown that the converse is also true (Put the word "Theorem" on top of the above conclusion, and then write $\Leftarrow$. The following part is to be said): if the spacetime metric admits a coordinate chart satisfying these four conditions, then the spacetime admits the special double foliation.

Proof. Given a coordinate chart $\{t, r, \theta, \phi\}$ of $(N, \overline{g})$ such that the $\overline{g}$ satisfies the four conditions, N is then foliated by t = constant slices which are spacelike since the metric has the form (4.2). For any t = constant slice, the coordinate spheres $\{S_{t,r}\}$ are solutions of a normal flow since $\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta} \rangle = \langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \phi} \rangle = 0$. We reparametrize the flow by setting $s := C + 2\ln r$, where C is a positive constant. Then

$$\frac{d}{dr}(dA) = \frac{d}{ds}(dA)\frac{ds}{dr} = \frac{2}{r}\frac{d}{ds}(dA). \tag{4.7}$$

On the other hand by condition (3)

$$\frac{d}{dr}(dA) = \frac{d}{dr}(r^2 \sin\theta d\theta d\phi) = 2r\sin\theta d\theta d\phi = \frac{2}{r}dA. \tag{4.8}$$

Combing the two equations above, we have $\frac{d}{ds}(dA) = dA$. Thus, by the first variation of area formula, $\{S_{t,r}\}$ when reparameterized by $r^2 = Ce^s$, are smooth solutions to inverse mean curvature flow. By condition (4), the mean curvature vector of $S_{t,r}$ stays tangential to the slice, therefore $\{S_{t,r}\}$ with the above reparameterization are smooth solutions to inverse mean curvature vector flow. □

Definition 4.1 (IMCVF Coordinate Chart)**.** *If a spacetime $(N^4, \overline{g})$ admits a coordinate chart $\{t, r, \theta, \phi\}$ such that the four conditions (4.3), (4.4), (4.5) and (4.6) are satisfied, then $\{t, r, \theta, \phi\}$ is called an IMCVF coordinate chart, and N is called a spacetime that admits an IMCVF coordinate chart.*

Thus, **finding spacetimes that have inverse mean curvature vector solutions is equivalent to finding spacetimes that admit inverse mean curvature vector flow coordinate charts.**

Many spherically symmetric spacetimes admit an IMCVF coordinate chart (e.g. coordinate chart (3.1) with $d = e = f = c = 0$, and $a = r^2$, $b = r^2\sin^2\theta$ and radial mean curvature vector by Equation (3.2)).

• Problem Statement: Given an arbitrary spacetime $(N^4, \overline{g})$, it is generally impossible to reparametrize it with an IMCVF coordinate chart (e.g. a spacetime that is not topologically equivalent to $(\mathbb{R}^3\backslash B_1) \times \mathbb{R}$). However, is it possible to *construct* a spacetime that admits such a coordinate chart?

Proposition 4.1 (Existence of Spacetimes That Admit an IMCVF Coordinate Chart)**.** *Let $U := (\mathbb{R}^3\backslash B_1) \times \mathbb{R} \subset \mathbb{R}^4$. There exist infinitely many spacetime metrics of the form of (4.2) that admits an IMCVF coordinate chart.*

Combining the above we have:

Theorem 4.1 (**Main Theorem**)**.** *There exist infinitely many non-spherically symmetric spacetimes that admit IMCVF coordinate charts. Given such a spacetime U with an IMCVF coordinate chart (t, r, θ, ϕ) and the constructed spacetime metric $\overline{g}$, the coordinate spheres $S_{t,r}$ contained in each $t =$ constant slice, when reparameterized by $r^2 = e^s$, are solutions to the smooth IMCVF equation.*

• Construction of Spacetime Metrics that Admit IMCVF Coordinate Charts: Let $U = (\mathbb{R}^3\backslash B_1) \times \mathbb{R}$. It is easy to construct a spacetime metric $\overline{g}$ that admits a coordinate chart $\{t, r, \theta, \phi\}$ that satisfies condition (4.3) and (4.4). Simply define $\overline{g}$ to have the form (4.2) (Point to the coordinate chart). Choosing two of the three variables a, b and c such that $ab - c^2 = r^4 \sin^2\theta$ satisfies condition (4.5).

The fourth condition requires $\vec{H}_{t,r}$, the mean curvature vector field of $S_{t,r}$, to be tangential to the $t =$ constant slice. This is equivalent to requiring $\vec{H}_{t,r}$ to be parallel to $\frac{\partial}{\partial r}$ (Keep this on the board along with $ab - c^2 = r^4 \sin^2\theta$). The normal bundle of $S_{t,r}$ has the following structure:

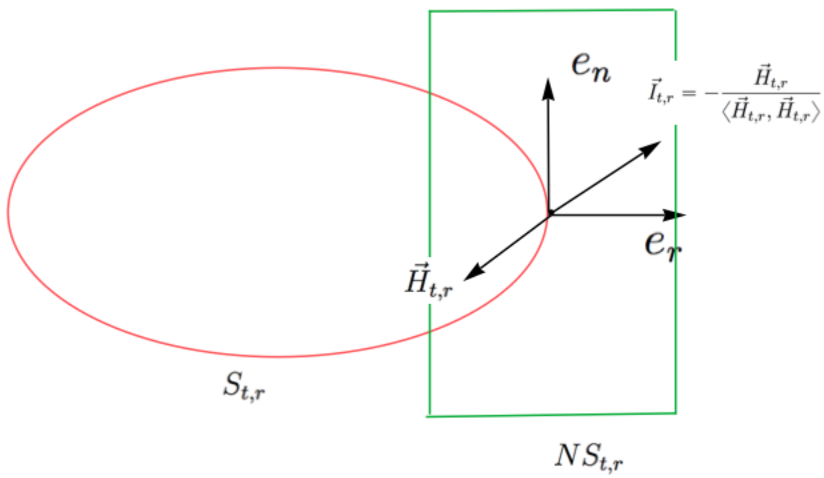

Figure 4: Normal bundle of coordinate sphere $S_{t,r}$.

where $e_r = \frac{1}{u}\frac{\partial}{\partial r}$, outward-spacelike unit normal, and e_n the complementary normal vector perpendicular to e_r. Therefore $\vec{H}_{t,r}$ can be written as

$$\vec{H}_{t,r} = \langle \vec{H}_{t,r}, e_r \rangle e_r - \langle \vec{H}_{t,r}, e_n \rangle e_n. \tag{4.9}$$

Thus, $\vec{H}_{t,r}$ parallel to $\frac{\partial}{\partial r}$ (Immediately write below it) is equivalent to (Use $\Longleftrightarrow$) $\langle \vec{H}_{t,r}, e_n \rangle = 0$. This equation can be computed out explicitly (see Section 4.3 of my thesis), and it is:

$$\begin{aligned}
0 &= \langle \vec{H}_{t,r}, e_n \rangle \\
&= \frac{1}{2|\overline{g}|}\Big\{ u^2(ab - c^2)[2be_{,\theta} - 2ce_{,\phi} - 2cf_{,\theta} + 2af_{,\phi}] \\
&\quad + d(ab - c^2)(ab - c^2)_{,r} + u^2(cf - be)[a_{,\theta}b - 2a_{,\phi}c + 2ac_{,\phi} - ab_{,\theta}] \\
&\quad + u^2(ce - af)[2bc_{,\theta} - a_{,\phi}b - 2b_{,\theta}c + ab_{,\phi}] \Big\}
\end{aligned}$$

What's amazing about this equation is that it is *zeroth order* in d. Thus we can choose all the other metric components (6 in total) (Point back to the metric (4.2) and identify the 6 free variables) and solve for d explicitly:

$$d = -\frac{u^2}{4r^3\sin^2\theta}\Big\{[2be_{,\theta} - 2ce_{,\phi} - 2cf_{,\theta} + 2af_{,\phi}] + \frac{cf - be}{r^4\sin^2\theta}[a_{,\theta}b - 2a_{,\phi}c + 2ac_{,\phi} - ab_{,\theta}] + \frac{ce - af}{r^4\sin^2\theta}[2bc_{,\theta} - a_{,\phi}b - 2b_{,\theta}c + ab_{,\phi}]\Big\}. \tag{4.10}$$

Therefore all four conditions (Point to the four conditions which should be kept on the board) can be solved with infinitely many solutions.

• Notice that (Point to Equation (4.10)) if all other variables are smooth, d will be also be smooth except possibly when $\sin\theta = 0$, since d is not defined by our formula there. Thus we have two *coordinate singularities* at the north and the south pole. d can be extended smoothly across these coordinate singularities if we choose c, e, f to be zero in a neighborhood of the north and south pole. This can be verified by looking at the explicit formula of d on page 51, Equation (4.2.28). One could study more general asymptotic conditions for d to be smooth and bounded as θ approaches 0 or π, but we choose not to discuss it further here.

• We have in total six degrees of freedom in constructing spacetime metrics with inverse mean curvature vector flow coordinates. Six is in fact the maximum possible degrees of freedom we can hope for, since a general reparameterization of a spacetime has six degrees of freedom. This greatly generalize the spherically symmetric case in two ways (Point to the spherically symmetric metric (3.1)):

(1) These six free variables do not need to be spherically symmetric: instead, they can depend on t, r, θ and ϕ;

(2) d, e, f and c do not need to be zero.

5 Steering Parameter and More Ways of Producing IMCVF Solutions: 3 minutes

• Given a spacetime $(N^4, \overline{g})$ and a spacelike hypersurface M, Given a closed surface Σ in M satisfying some natural expansion conditions (Draw Figure 5).

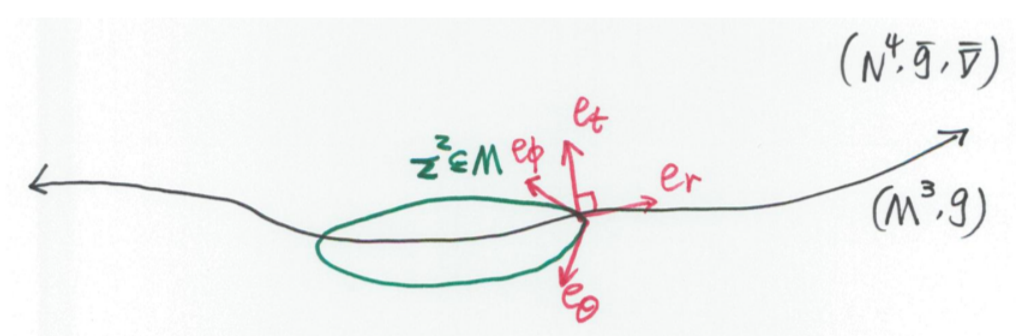

Figure 5: IMCVF steering setup

We showed that one can change the one component of the spacetime metric, called a *steering*, and the mean curvature vector of Σ will be tangential to M in the steered metric. This can be used to generate more solutions of inverse mean curvature vector flow: Consider a smooth solution to the inverse mean curvature flow (Draw a foliation of IMCF in M), we can then smoothly adjust the spacetime metric such that the mean curvature vector of each flow surface becomes tangential to M. These steered surfaces are then solutions to inverse mean curvature vector flow, since the area expanding condition is already satisfied, and now the mean curvature vectors are tangential to a spacelike slice.

6 Conclusions and Open Problems: 4 minutes

Problem 1. *We have constructed many examples of spacetimes that admit smooth inverse mean curvature vector flow solutions. These examples suggest that given a spacetime, it might be possible in general to find the "right" initial surface such that the inverse mean curvature vector flow exists.*

Going to the big picture of relating local and global notions of mass, it is still unknown that:

Problem 2. *(You can simply state this without writing it down, simple draw the local to global mass picture again) Given a spacetime that is sufficiently asymptotically flat, e.g. Schwarzschild outside a compact set, then does the Hawking of inverse mean curvature vector flow surfaces approach the total mass of the spacetime?*

A natural next step is to consider the following questions:

Problem 3. *Given a spherically symmetric spacetime that admits an inverse mean curvature vector flow coordinate chart. Consider a perturbation of the spacetime metric. Does the perturbed metric admit an inverse mean curvature vector flow coordinate chart as well?*

We conjecture that this is always the case for some Minkowski spacetimes:

Conjecture 6.1. *Given Minkowski space with IMCVF coordinate chart that can be smoothly extended to the boundary, consider a perturbation of the spacetime metric. The resulting spacetime still admits IMCVF solutions (in a single spacelike hypersurface) that exist for all time.*

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