

THE MOTIVE OF THE FANO SURFACE OF LINES

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ABSTRACT. The purpose of this note is to prove that the motive of the Fano surface of lines on a smooth cubic threefold is finite-dimensional in the sense of Kimura.

1. INTRODUCTION

Let k be a field with $\text{char } k \neq 2$ and let $X \subset \mathbb{P}^4$ be a smooth cubic threefold. We denote by $S(X)$ (or just S) the Fano variety of lines in X . This is known to be a smooth, connected projective surface of general type. It turns out that this surface possesses a great many remarkable properties. It is known, for instance, that the Albanese map is an imbedding $i : S \hookrightarrow A := \text{Alb}(S)$ and that the pull-back $H^2(A) \xrightarrow{i^*} H^2(S)$ is an isomorphism. In this note, we will prove a motivic version of this isomorphism. For this, let \mathcal{M}_k denote the category of pure Chow motives with rational coefficients and let $\mathfrak{h} : \mathcal{V}_k^{\text{opp}} \mapsto \mathcal{M}_k$ be the functor that sends a smooth projective k -variety X to $\mathfrak{h}(X) = (X, \Delta_X, 0) \in \mathcal{M}_k$. Then, our first result is that:

Theorem 1.1. *The morphism $\mathfrak{h}(i) : \mathfrak{h}(A) \rightarrow \mathfrak{h}(S)$ is split-surjective.*

This will show that the motive of the Fano surface is *finite-dimensional in the sense of Kimura*. The primary known examples of surfaces with finite-dimensional motive are those for which either:

- (i) The Chow group of nullhomologous 0-cycles is representable.
- (ii) There exists a dominant rational map from a product of smooth projective curves.

Since $p_g(S) > 0$, Mumford's Theorem ([15] Theorem 3.13) shows that $CH_{\text{hom}}^2(S)$ is not representable. Moreover, a recent result in [14] shows that S is not dominated by a product of curves. To the author's knowledge, the Fano surface is the first example of a surface with finite-dimensional motive for which neither (i) nor (ii) holds. One reason for interest in finite-dimensionality is that if $M \in \mathcal{M}_k$ is finite-dimensional, then any morphism $f : M \rightarrow M$ that induces an isomorphism on cohomology is actually an isomorphism of motives. This will be important to proving:

Theorem 1.2. *The pull-back along the Albanese imbedding $i : S \hookrightarrow A$ induces an isomorphism $\mathfrak{h}^2(A) \xrightarrow{\cong} \mathfrak{h}^2(S)$ in \mathcal{M}_k .*

The plan will be as follows. We will give a proof of the Theorem 1.1, then review the notion of finite-dimensionality for motives which will allow us to prove Theorem 1.2. In this note, all Chow groups will be taken with rational coefficients.

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2. A GENERAL PRINCIPLE

The following variant of the Manin principle will facilitate the proof of Theorem 1.1, as pointed out by B. Kahn.

Proposition 2.1. *Let X and Y be smooth projective connected varieties over a field k and $f \in \text{Cor}^0(X, Y)$ such that $f_{\Omega*} : CH^i(X_{\Omega}) \rightarrow CH^i(Y_{\Omega})$ is surjective for all i and every algebraically closed extension $k \subset \Omega$. Then, f is split-surjective.*

This principle has appeared in the literature in various guises (see, for instance, [7] Theorems 3.5 and 3.6). However, since this exact statement is difficult to find, we give a proof below, following [8]. We subdivide the proof into three claims:

Claim: $f_{L*} : CH^i(X_L) \rightarrow CH^i(Y_L)$ is surjective for all i and every extension $k \subset L$.

Proof of Claim. Let $\beta \in CH^i(Y_L)$. We need to find $\alpha \in CH^i(X_L)$ for which $f_{L*}(\alpha) = \beta$. By assumption, $f_{\bar{L}*} : CH^i(X_{\bar{L}}) \rightarrow CH^i(Y_{\bar{L}})$ is surjective, so there is some $\bar{\alpha} \in CH^i(X_{\bar{L}})$ such that $f_{\bar{L}*}(\bar{\alpha}) = \beta_{\bar{L}}$. Let $L' \supset L$ be some finite Galois extension for which there is $\alpha' \in CH^i(X_{L'})$ with $\alpha'_{\bar{L}} = \bar{\alpha}$. Indeed, there is some finite extension for which this holds; since Chow groups are unchanged by passing to a purely inseparable extension, we can assume that L' is a separable extension. This allows us to pass to a Galois extension. Then, let $G = \text{Gal}(L'/L)$ and define $\text{Tr}(\alpha') = \frac{1}{|G|} \sum_{g \in G} g^* \alpha'$. By [5] Example 1.7.2.6, this yields $\alpha \in CH^i(X_L)$ such that $\alpha_{L'} = \text{Tr}(\alpha')$. We compute:

$$(f_{L*}(\alpha))_{L'} = f_{L'*}(\alpha_{L'}) = f_{L'*}(\text{Tr}(\alpha')) = \text{Tr}(f_{L'*}(\alpha')) = \text{Tr}(\beta_{L'}) = \beta_{L'}$$

Since $CH^i(X_L) \rightarrow CH^i(X_{L'})$ is injective, it follows that $f_{L*}(\alpha) = \beta$, as desired. \square

Claim: $(f \times \text{id}_Z)_* : CH^i(X \times Z) \rightarrow CH^i(Y \times Z)$ is surjective for all i and all smooth projective connected varieties Z over k .

Proof of Claim. We proceed by induction on $n = \dim(Z)$. The case $n = 0$ is true by the preceding claim. Assume then that the claim holds for $n - 1$. To prove it for n , we use the

following commutative diagram with rows exact:

$$(1) \quad \begin{array}{ccccccc} \varinjlim_W CH^{i-c}(X \times W) & \xrightarrow{(id_X \times j_W)_*} & CH^i(X \times Z) & \longrightarrow & CH^i(X_{k(Z)}) & \longrightarrow & 0 \\ (f \times id_W)_* \downarrow & & (f \times id_Z)_* \downarrow & & (f_{k(Z)})_* \downarrow & & \\ \varinjlim_W CH^{i-c}(Y \times W) & \xrightarrow{(id_Y \times j_W)_*} & CH^i(Y \times Z) & \longrightarrow & CH^i(Y_{k(Z)}) & \longrightarrow & 0 \end{array}$$

where $k(Z)$ is the function field of Z , the limit ranges over smooth projective varieties W with $\dim(W) \leq \dim(Z)$ and morphisms $j_W : W \rightarrow Z$ and $c = \dim(Z) - \dim(W)$. The commutativity of the diagram is immediate. Once we have established the exactness of the rows, we see that the right vertical arrow is surjective by the preceding claim. Moreover, the surjectivity of the left vertical arrow follows by the inductive hypothesis. Then, a diagram chase shows that the middle vertical arrow is surjective also.

What remains is to show that the rows are exact. To this end, note that the rows differ from the usual localization sequences in that the varieties W are required to be smooth (and are not required to be subvarieties of Z). By de Jong's theorem on alterations, for any closed subvariety $V \subset Z$, there exists some smooth projective W and some generically finite morphism $p : W \rightarrow V$. So, it suffices to show that the push-forward

$$CH^*(X \times V) \xrightarrow{(id_X \times p)_*} CH^*(X \times W)$$

is surjective. This is a straightforward exercise in induction which is left to the reader. \square

Claim: f possesses a right-inverse.

Proof of Claim. Taking $Z = Y$ and $i = \dim(Y)$ in the above claim, we obtain that

$$CH^i(Y \times X) \xrightarrow{(id_Y \times f)_*} CH^i(Y \times Y)$$

is surjective. So, there is some $\gamma \in CH^i(X \times Y)$ for which $(id_Y \times f)_* \gamma = \Delta_Y$. Applying Lieberman's lemma ([5] Proposition 16.1.1) then gives

$$f \circ \gamma = (id_Y \times f)_* \gamma = \Delta.$$

This is the desired result. \square

3. PROOF OF THEOREM 1.1

Assume that S has a k -rational point and let $i : S \rightarrow A$ be the corresponding Albanese morphism. To prove the theorem, we need to show that the correspondence ${}^t\Gamma_i \in \text{Cor}^0(A \times S)$ possesses a right-inverse. The general principle then shows that it suffices to prove that the pull-back $i_\Omega^* : CH^j(A_\Omega) \rightarrow CH^j(S_\Omega)$ is surjective for $j = 1, 2$ and all algebraically closed extensions $k \subset \Omega$.

For a smooth cubic threefold X over k , S is the Hilbert scheme of X with Hilbert polynomial

$t + 1$. We recall the following base change compatibility for Hilbert schemes (see, for instance, [12]); i.e., for an extension $k \subset \Omega$, we have

$$S(X_\Omega) \cong S(X)_\Omega$$

Thus, S_Ω is the Fano surface of lines of X_Ω . Moreover, since the Albanese is compatible with base extension, we view $i_\Omega : S_\Omega \rightarrow A_\Omega$ as an Albanese map.

Proposition 3.1. *For all algebraically closed extensions $k \subset \Omega$, the pull-back $i_\Omega^* : CH^1(A_\Omega) \rightarrow CH^1(S_\Omega)$ is an isomorphism.*

Proof. Since $i_\Omega : S_\Omega \rightarrow A_\Omega$ is an Albanese map, we have $Pic^0(A_\Omega) \xrightarrow{i_\Omega^*} Pic^0(S_\Omega)$ is an isomorphism, and so it suffices to show that we have an isomorphism $i_\Omega^* : NS(A_\Omega)_\mathbb{Q} \rightarrow NS(S_\Omega)_\mathbb{Q}$ of Néron-Severi groups. Now, let $\ell \neq \text{char } k$. Then, $H^2(A_\Omega, \mathbb{Q}_\ell(1)) \xrightarrow{i_\Omega^*} H^2(S_\Omega, \mathbb{Q}_\ell(1))$ is an isomorphism by [13] Proposition 4. Note that S and A may be defined over some finitely generated field k_0 . Upon passing to a large enough extension of k_0 , we may also assume that C possesses a model over k_0 and that $NS(C_{k_0} \times A_{k_0})_\mathbb{Q} \cong NS(C \times A)_\mathbb{Q}$ (and, similarly, for $C \times S$). Note that this is possible because the Néron-Severi group is finitely generated. Thus, we need to show that

$$NS(C_{k_0} \times A_{k_0})_\mathbb{Q} \xrightarrow[\cong]{(id_C \times i)^*} NS(C_{k_0} \times S_{k_0})_\mathbb{Q}$$

For this, let $G := \text{Gal}(k/k_0)$ be the absolute Galois group and $\ell \neq \text{char } k$. The Künneth theorem on cohomology then gives a (G -module) isomorphism:

$$(2) \quad H^2(C \times A, \mathbb{Q}_\ell(1)) \xrightarrow[\cong]{(id_C \times i)^*} H^2(C \times S, \mathbb{Q}_\ell(1))$$

The result then follows from applying the functor $H^0(G, -)$ to (2) and noting that the Tate conjecture holds for the left-hand side (by Faltings' theorem and the fact that k_0 is a finitely generated field). \square

We have the following important result of Bloch:

Proposition 3.2 ([3] 1.7). *Let S be the Fano surface of lines of a smooth cubic threefold in \mathbb{P}^4 over an algebraically closed field of characteristic $\neq 2$. Then, the intersection product $CH^1(S) \otimes CH^1(S) \rightarrow CH^2(S)$ is surjective.*

Proposition 3.3. *For all algebraically closed extensions $k \subset \Omega$, the pullback $CH^2(A_\Omega) \xrightarrow{i_\Omega^*} CH^2(S_\Omega)$ is surjective.*

Proof. Since pull-back commutes with intersection product on Chow groups, we have the following diagram:

$$(3) \quad \begin{array}{ccc} CH^1(A_\Omega) \otimes CH^1(A_\Omega) & \xrightarrow{\quad \cdot \quad} & CH^2(A_\Omega) \\ i_\Omega^* \times i_\Omega^* \downarrow & & i_\Omega^* \downarrow \\ CH^1(S_\Omega) \otimes CH^1(S_\Omega) & \xrightarrow{\quad \cdot \quad} & CH^2(S_\Omega) \end{array}$$

From Corollary ??, the left vertical arrow is surjective. Since Ω is algebraically closed and S_Ω is the Fano surface of lines of X_Ω , we can apply Proposition 3.2 to deduce that the bottom horizontal arrow is also surjective.. So, the right vertical arrow is surjective, as desired. \square

4. FINITE-DIMENSIONALITY OF MOTIVES

This section reviews the definition and properties of finite-dimensionality before proving Theorem 1.2. Recall that the category of Chow motives \mathcal{M}_k is a tensor category with tensor product defined as:

$$(4) \quad (X, \pi, m) \otimes (Y, \tau, n) := (X \times Y, \pi \times \tau, m + n)$$

We can define an action of the symmetric group $\mathbb{Q}[\mathfrak{S}_n] \rightarrow \text{End}_{\mathcal{M}_k}(M^{\otimes n})$ for $M \in \mathcal{M}_k$. Since \mathcal{M}_k is a pseudo-Abelian category, all idempotents possess images in \mathcal{M}_k . So, for any idempotent in the group algebra $\mathbb{Q}[\mathfrak{S}_n]$, there is a corresponding motive. In particular, we have

$$(5) \quad \begin{aligned} \text{Sym}^n M &= \text{Im}(\pi_{\text{sym}}) \\ \wedge^n M &= \text{Im}(\pi_{\text{alt}}) \end{aligned}$$

for the symmetric and the alternating representation of \mathfrak{S}_n .

Definition 4.1 (Kimura). *A motive $M \in \mathcal{M}_k$ is said to be oddly finite-dimensional if $\text{Sym}^n M = 0$ for $n \gg 0$ and evenly finite-dimensional if $\wedge^n M = 0$ for $n \gg 0$. M is said to be finite-dimensional if $M = M_+ \oplus M_-$, where M_+ is evenly finite-dimensional and M_- is oddly finite-dimensional.*

We have the following properties of finite-dimensional motives:

Theorem 4.1 (Kimura, [9]).

- (a) *The motive of a smooth projective curve is finite-dimensional.*
- (b) *If $M, N \in \mathcal{M}_k$ are finite-dimensional, then so are $M \oplus N$ and $M \otimes N$. Conversely, if $M \oplus N$ is finite-dimensional, then so are M and N .*
- (c) *If $f : M \rightarrow N$ is split-surjective and M is finite-dimensional, then so is N .*
- (d) *If M is finite-dimensional and the odd degree cohomology $H^-(M) = 0$ for some Weil cohomology H^* , then M is evenly finite-dimensional (and, similarly for oddly finite-dimensionality).*

(e) Suppose M is evenly (or oddly) finite-dimensional. Then, for every Weil cohomology H^* , the map $\text{End}_{\mathcal{M}_k}(M) \rightarrow \text{End}(H^*(M))$ is injective.

As a consequence of (a), (b), the motive of any product of smooth projective curves is finite-dimensional; from (c), so is any variety dominated by a product of curves (such as Abelian varieties). In [6] it is demonstrated that varieties of dimension ≤ 3 have finite-dimensional motive if $CH_0(X)_{\text{hom}}$ is representable. (This is true, in particular, if X is a rationally connected threefold.) Other than this, the following conjecture remains wide open.

Conjecture 4.1 (Kimura, O’Sullivan). *Every motive $M \in \mathcal{M}_k$ is finite-dimensional.*

To prove Theorem 1.2, we will need the next two results.

Proposition 4.1 ([11] Theorem 3). *Let X be a smooth projective surface. Then, there are idempotents $\pi_i \in \text{End}_{\mathcal{M}_k}(\mathfrak{h}(X))$ satisfying the following conditions:*

- (i) $\Delta_X = \pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4$
- (ii) $\pi_i \circ \pi_j = 0$ for $i \neq j$
- (iii) *The motive $\mathfrak{h}^i(X) = (X, \pi_i, 0)$ satisfies $H^*(\mathfrak{h}^i(X)) = H^i(X)$ for any Weil cohomology H^* .*

Remark 4.1. *It should be noted that the idempotents in Proposition 4.1 are not unique. Some extra conditions (see loc. cit.) may be imposed on π_i so that the resulting motives are unique up to isomorphism.*

When X is an Abelian variety of dimension g , we have the following result of Denninger and Murre which was proved using Beauville’s Fourier transform ([2]).

Theorem 4.2 ([4] Theorem 3.1). *There is a unique decomposition*

$$(6) \quad \mathfrak{h}(X) = \bigoplus_{i=0}^{2g} \mathfrak{h}^i(X)$$

where $\mathfrak{h}^i(X) = (X, \pi_i, 0)$ with π_i idempotents satisfying:

- (i) $\pi_i \circ \pi_j = 0$ for $i \neq j$;
- (ii) ${}^t\Gamma_n \circ \pi_i = n^i \cdot \pi_i = \pi_i \circ {}^t\Gamma_n$ for all $n \in \mathbb{Z}$;
- (iii) ${}^t\pi_i = \pi_{2g-i}$.

Remark 4.2. *As suggested by the notation, we have $H^*(\mathfrak{h}^i(X)) = H^i(X)$.*

Since the motive of an Abelian variety X is finite-dimensional and the cohomology of $\mathfrak{h}^i(X)$ is concentrated in one degree, Theorem 4.1 implies that $\mathfrak{h}^i(X)$ is finite-dimensional of parity $i \pmod{2}$. From Theorem 1.1 and Theorem 4.1 (c), it follows that the motive of the Fano surface $\mathfrak{h}(S)$ is also finite-dimensional and the summand $\mathfrak{h}^2(S)$ as in Proposition 4.1 is evenly finite-dimensional.

Proof of Theorem 1.2. Fix $\pi_{2,S}$ as in Proposition 4.1. We define

$$\mathfrak{h}^2(i) = \pi_{2,S} \circ {}^t\Gamma_i \circ \pi_{2,A} \in \mathrm{Hom}_{\mathcal{M}_k}(\mathfrak{h}^2(A), \mathfrak{h}^2(S))$$

The goal is to show that $\mathfrak{h}^2(i)$ is an isomorphism of motives. This means that we need to find some $\psi \in \mathrm{Hom}_{\mathcal{M}_k}(\mathfrak{h}^2(S), \mathfrak{h}^2(A))$ for which

$$(7) \quad \psi \circ \mathfrak{h}^2(i) = \pi_{2,A} \in \mathrm{End}_{\mathcal{M}_k}(\mathfrak{h}^2(A)), \quad \mathfrak{h}^2(i) \circ \psi = \pi_{2,S} \in \mathrm{End}_{\mathcal{M}_k}(\mathfrak{h}^2(S))$$

Since $\mathfrak{h}^2(A)$ and $\mathfrak{h}^2(S)$ are evenly finite-dimensional, it suffices by Theorem 4.1 (e) to find some such ψ for which these equalities hold cohomologically. Moreover, $i^* = \mathfrak{h}^2(i)_* : H^2(A, \mathbb{Q}_\ell) \rightarrow H^2(S, \mathbb{Q}_\ell)$ is an isomorphism for $\ell \neq \mathrm{char} k$. Thus, if we can find some

$$(8) \quad \gamma \in CH^2(S \times A)$$

for which $\gamma_* : H^2(S, \mathbb{Q}_\ell) \rightarrow H^2(A, \mathbb{Q}_\ell)$ is the inverse of i^* , we can set $\psi := \pi_{2,A} \circ \gamma \circ \pi_{2,S}$, and this gives the desired correspondence in (7).

From [1], the image of the map $S \times S \rightarrow A$ defined by $(x, y) \mapsto i(x) - i(y)$ is an ample divisor Θ . Moreover, we have

$$\frac{1}{3!} \cdot \Theta \wedge \Theta \wedge \Theta = [S] \in H^6(A, \mathbb{Q}_\ell(3))$$

The Hard Lefschetz theorem then implies that $\wedge[S] : H^2(A, \mathbb{Q}_\ell) \rightarrow H^8(A, \mathbb{Q}_\ell)(3)$ is an isomorphism. Also, since the Lefschetz standard conjecture is true for Abelian varieties ([10] Proposition 4.3), it follows that there is a correspondence $\phi \in CH^2(A \times A)$ such that $\phi_* : H^8(A, \mathbb{Q}_\ell) \rightarrow H^2(A, \mathbb{Q}_\ell)(-3)$ is the inverse of $\wedge[S]$. The projection formula then shows that $i_* \circ i^* = \wedge[S]$ so that $\phi_* \circ i_* \circ i^*$ is the identity on $H^2(A, \mathbb{Q}_\ell)$. Since $i^* : H^2(A, \mathbb{Q}_\ell) \rightarrow H^2(S, \mathbb{Q}_\ell)$ is an isomorphism, it follows that $\gamma = \phi \circ \Gamma_i$ is the desired inverse.

□

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